Cantor Set and Dimension: Additional Problems

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1 More Exercises on Fractal Dimensions

Recall that in the last handout, we defined $K_t(X)$ and $P_t(X)$ to be the least number of intervals of length t required to cover a set X in the real line, and the largest number of intervals that could be backed disjointly into X respectively. From these, we defined a *covering* or *packing* dimension Now we will use higher dimensional shapes to cover and pack X in two dimensions. If the shape is Y, $K_m(X)$ will be the least number of copies of Y, scaled to have measure (length, area, or volume) m, required to cover X, and $P_m(X)$ will be the largest number of non-overlapping copies of Y, scaled to have measure m, that can be placed with non-empty intersection with X.

The packing/covering dimension is defined in d dimensions by

$$d\frac{\log P_m(X)}{|\log(m)|} \approx d\frac{\log K_m(X)}{|\log(m)|}$$

for very small m. This turns out not to depend very much on the choice of shape Y, but we will not prove that here.

Problem 1 The Koch snowflake is the curve created by following this recursive process for infinitely many steps:

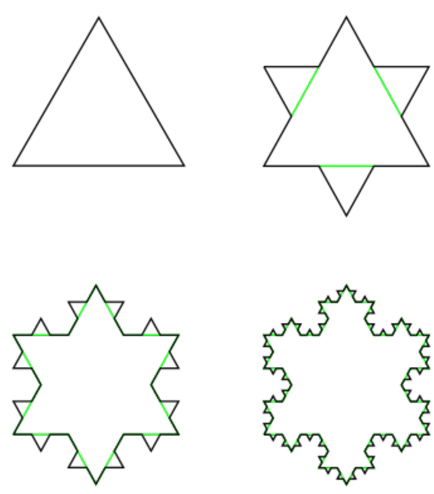
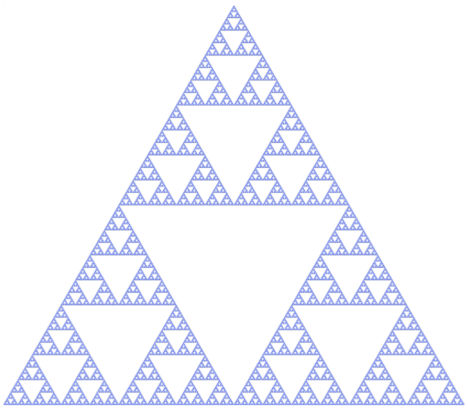


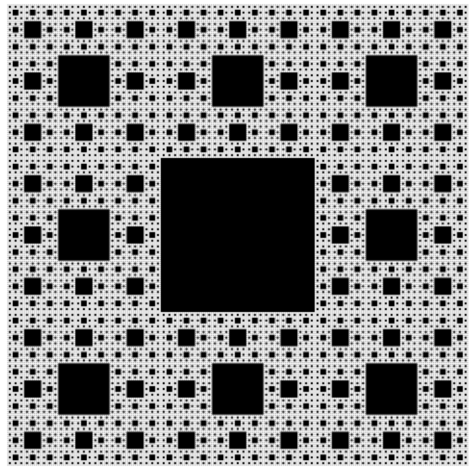
Figure 1: Koch Snowflake `https://en.wikipedia.org/wiki/File:KochFlake.svg`

Compute the area of the interior of this curve, and the length of the nth stage in the construction of the curve. Using an equilateral triangle for Y, compute the covering and packing dimensions of the curve (just the boundary).

Problem 2 Compute the covering and packing dimensions of the Sierpinski triangle and Sierpinski carpet with equilateral triangles and squares respectively:



(a) Sierpinski Triangle https://commons.wikimedia.org/wiki/File:Sierpinski_triangle.svg



(b) Sierpinski Carpet (The white region, not the black) https://en.wikipedia.org/wiki/File:Sierpinski_carpet_6.svg

Problem 3 Now jumping to 3 dimensions, try using cubes to compute the covering and packing dimensions of the Menger sponge, which is constructed by iterating this process:

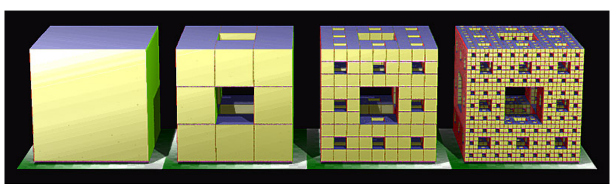


Figure 3: Menger Sponge https://commons.wikimedia.org/wiki/File:Menger_sponge_(Level_0-3).jpg

2 The Cantor Function

Here we seek to define a function $g : [0, 1] \to [0, 1]$ in terms of ternary and binary expansions.

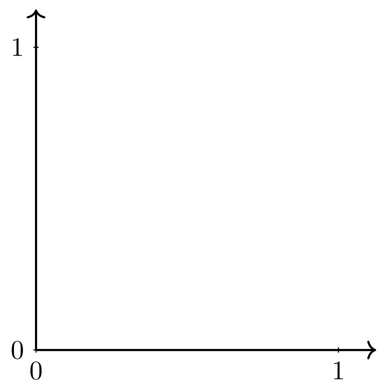
First we define a function f on the Cantor set C_∞. If $x \in C_\infty$ has the ternary expansion $0.a_1a_2\dots$ (with only 0s and 2s), then we define $f(x)$ to be the *binary* number $0.b_1b_2\dots$, where $b_i = a_i/2$ for each i.

Problem 4 Prove that this definition is well-defined, in that each $x \in C_\infty$ has a unique ternary expansion with only 0s and 2s, so there isn't ambiguity in choosing which sequence to use to define $f(x)$.

Problem 5 Prove that $f : C_\infty \to [0,1]$ is (not necessarily strictly) increasing, and is surjective (onto). Note that this means that the measure-0 C_∞ can be stretched around in ways that respect its order to cover the measure-1 $[0,1]$!

Problem 6 Prove that there is a unique way to extend f to a (not necessarily strictly) increasing function $g : [0,1] \to [0,1]$. (By extend, we mean that if $x \in C_\infty$, $g(x) = f(x)$.)

Problem 7 Graph g here:



Hint: because C_∞ has measure 0, it is essentially invisible, so you can focus on graphing the function on the intervals between C_∞.

Depending on the definition of continuous you prefer to use, conclude (possibly just by looking at the graph) that g is continuous. This means that it is possible for a function which is (piecewise) constant on all but a measure 0 set can still change enough to cover actual distance – moving from $g(0) = 0$ to $g(1) = 1$ – without any sudden jumps!