# Graph Theory and Some Topology 2

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## 1 More on Brouwer's Fixed Point Theorem

**Problem 1** As a group, discuss why this formal statement and this casual interpretation of Brouwer's Fixed Point Theorem are the same:

- If $f : \Delta \to \Delta$ is continuous, then there exists $x \in \Delta$ with $f(x) = x$.
- If you take a piece of paper and lay it on a table, and then crumple an identical piece of paper and put it on top of the first piece of paper, some point on the crumpled paper corresponds exactly to the point on the flat paper directly below it.

**Problem 2** Treat each of these subsets of $\mathbb{R}^2$ as a metric space with the standard Euclidean distance. Does each one satisfy the result of Brouwer's Fixed Point Theorem? That is, if the space is $X$, does a continuous function $f : X \to X$ necessarily have to have a fixed point?

- The unit circle, defined by $x^2 + y^2 = 1$
- The closed unit disc, defined by $x^2 + y^2 \leq 1$
- The open unit disc, defined by $x^2 + y^2 < 1$

**Problem 3** Using Brouwer's Fixed Point Theorem, show that there does not exist a continuous function $f : D \to C$, where $D$ is the closed unit disc, and $C$ is its boundary, the unit circle, that fixes the circle pointwise, that is, $f(x) = x$ for all $x$ on the circle.

## 2 Euler Characteristic of Surfaces

The Euler characteristic of any surface can be defined by replacing its curved sections with similarly-shaped polygons, and then calculating the Euler characteristic of the resulting polyhedral surface, as $|V| - |E| + f$, where $V, E$ are the vertex and edge sets of the graph, and $f$ is the number of polygonal faces the graph breaks the surface into. In the case where the surface is the plane, this is our original definition with planar graphs, with the curved sections already flat. We've also calculated the Euler characteristic of the sphere, by modifying it into any of the Platonic solids.

**Problem 4**

**(a)** What is the Euler characteristic of the closed disc, given by $x^2 + y^2 \leq 1$ in $\mathbb{R}^2$?

**(b)** Find a polyhedral version of the torus, and use it to calculate the Euler characteristic of the torus.

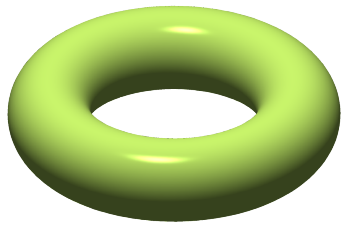

Figure 1: The torus[1]

## 2.1 Surgery

Imagine taking two copies of the torus, cutting a small circular hole out of each one, and then gluing the surfaces together at the boundaries of the circular holes, to get a double torus:

Figure 2: The double torus[2]

This process, known as "surgery," can also be done with two polyhedral surfaces. Given a pair of faces, one on each surface, with the same number of edges, we can remove the faces and then glue together the vertices and edges that were the boundaries of those faces. This reduces the number of faces by simply removing them, and reduces the number of vertices and the number of edges by merging together existing vertices and edges.

**Problem 5**

**(a)** What is the Euler characteristic of the double torus?

**(b)** Let $P_1, P_2$ be two polyhedral surfaces, with Euler characteristics $\chi_1, \chi_2$. If we do surgery on $P_1$ and $P_2$, by removing an $n$-gon face from each one and gluing together their boundaries, what is the Euler characteristic of the resulting figure?

**(c)** What's the Euler characteristic of the triple torus? How about an $n$-tuple torus?

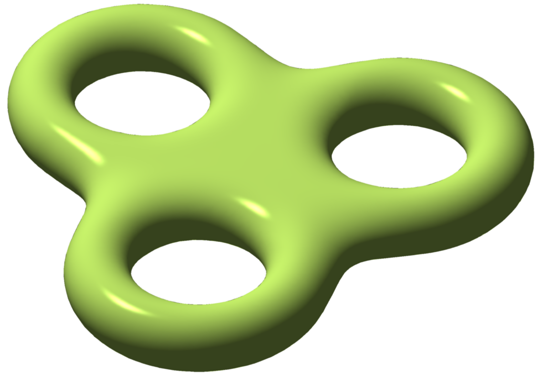

Figure 3: The triple torus[3]

# 3 More on Planarity

Define $K_n$ to be the *complete* graph on $n$ vertices: the graph such that all vertices are connected. Also define $K_{m,n}$ to be the *complete bipartite* graph on $m$ and $n$ vertices: a graph with its $m+n$ vertices split into two parts, $V_1, V_2$, where $|V_1| = m, |V_2| = n$, and two vertices are connected if and only if they belong to different parts.

**Problem 6**

**(a)** For which $n$ is $K_n$ planar?

**(b)** A *bipartite* graph is any subgraph of some $K_{m,n}$. Show that if a graph is bipartite, it contains no cycles of odd length.

**(c)** For which $m, n$ is $K_{m,n}$ planar?

## 3.1 Crossing Number

Let the *crossing number* $\mathrm{cr}(G)$ of a graph $G$ be the minimum number of pairs of edges that need to cross in order to draw $G$ in the plane.

**Problem 7**

**(a)** Show that $\mathrm{cr}(G) = 0$ if and only if $G$ is planar.

**(b)** Show that $\mathrm{cr}(K_5) = \mathrm{cr}(K_{3,3}) = 1$.

**Problem 8** Show that $\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$. (Hint: Look at the contribution from each $K_5$-shaped subgraph.) It is known that $\mathrm{cr}(K_n) \leq \frac{3}{8}\binom{n}{4}$, so your lower bound is correct within a factor of 2.

**Problem 9** Using results about planar graphs from last week, show that $\mathrm{cr}(G) \geq |E| - 3|V|$ for any graph $G$ with vertices $V$ and edges $E$.

**Problem 10** Let $G$ be a graph where $|E| > 4|V|$. We will use probability to show that

$$\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}.$$

Say you have a loaded coin, which comes up heads with probability $p$. Now for each vertex $v \in V$, flip the coin, and put $v$ in the set $V_H$ if the coin comes up heads. Now let $H$ be the graph with vertex set $V_H$, where $v, w \in V_H$ are connected with an edge if and only if they are in $G$.

**(a)** What is the probability that a given edge $e$ of $G$ is in $H$?

**(b)** Asssume $G$ is drawn in the plane with exactly $\mathrm{cr}(G)$ crossings, and $H$ is drawn the same way, except with some vertices and edges missing. If $e_1, e_2$ are edges of $G$ that cross, what is the probability that both are in $H$?

**(c)** Recall that if $X$ is a random variable, $\mathbb{E}[X]$ denotes the *expectation* of $X$, or the average value it takes. Find the expectation of these three variables:

- $|V_H|$, the number of vertices of $H$
- $|E_H|$, the number of edges of $H$
- $c_H$, The number of crossings in the drawing of $H$

**(d)** Based on the idea of linearity of expectation, explain why $\mathbb{E}[c_H] \geq \mathbb{E}[|E_H|] - 3\mathbb{E}[|V_H|]$.

**(e)** Set $p = \frac{4|V|}{|E|}$. Convince yourself that this is a valid probability. Then combine the last two parts of this problem and prove the inequality

$$\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}.$$

**Problem 11** Let $G$ be a graph with $n$ vertices each of degree at least 9. Show that the crossing number of $G$ is at least $\frac{4n}{3}$.

# References


[1] URL: `https://en.wikipedia.org/wiki/File:Torus_illustration.png`.

[2] URL: `https://en.wikipedia.org/wiki/File:Double_torus_illustration.png`.

[3] URL: `https://en.wikipedia.org/wiki/File:Triple_torus_illustration.png`.