Dynamical Systems

Aaron Anderson

January 2021

1 Cobweb Plots

Let f be a real-valued function such that if $0 \leq x \leq 1$, then $0 \leq f(x) \leq 1$. To draw a *cobweb plot* of f, first plot the functions $y = x$ and $y = f(x)$. Pick a starting value $x_0 \in [0, 1]$. Start by plotting a line from $(x_0, 0)$ to $(x_0, f(x_0))$, then draw a line from $(x_0, f(x_0))$ to $(f(x_0), f(x_0))$. Keep connecting dots, alternating between vertical lines (x, x) to $(x, f(x))$ and horizontal lines $(x, f(x))$ to $(f(x), f(x))$.

The description's a bit clunky, but here's an example:

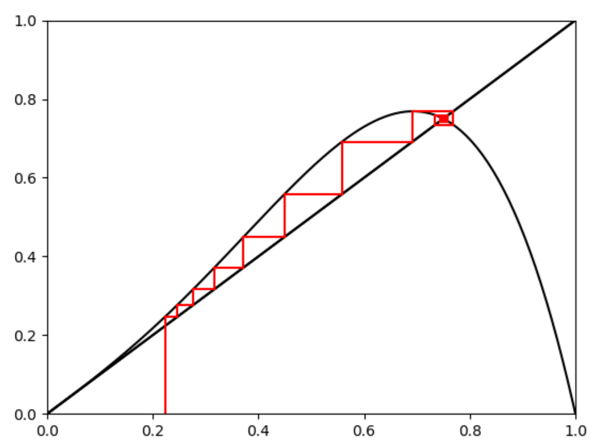
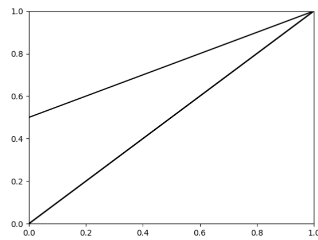


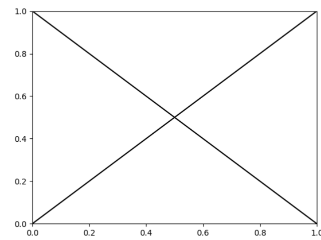
Figure 1: Cobweb plot for $f(x) = x(1-x)(4x^2 + x + 1)$, with $x_0 = 0.223$

Problem 1. Let $f(x) = x(1-x)(4x^2 + x + 1)$, $x_0 = 0.223$. What does the cobweb plot tell us about the sequence $x_0, f(x_0), f(f(x_0)), \ldots$?

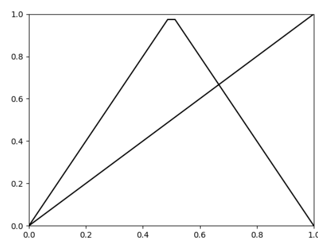
Problem 2. Fill out the following cobweb plots with starting point $x_0 = \frac{1}{5}$:



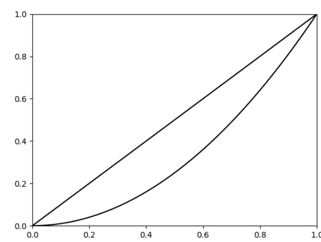
(a) $f(x) = \frac{x+1}{2}$



(b) $f(x) = 1 - x$



(c) $f(x) = 2 - 2|x|$



(d) $f(x) = x^2$

Definition 1.1. We say that a fixed point x_0 of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is *stable* if there is some interval (a, b) containing x_0 such that if $x \in (a, b)$, then the sequence $x, f(x), f(f(x)), \ldots$ converges to x_0. Otherwise it is called *unstable*.

Problem 3. Let $f(x) = ax + b$ be a linear function. For what values of a, b does f have a fixed point, and when does f have a stable fixed point?

The following challenge problem shows that stable fixed points are the only situations where the sequence $x, f(x), f(f(x)), \ldots$ converges.

Problem 4. Challenge: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that if a sequence $x_0, x_1, x_2, \ldots$ satisfying $x_{n+1} = f(x_n)$ converges to a number x_∞, then $f(x_\infty) = x_\infty$.

2 Logistic Maps

In this section, we study some functions called the *logistic map*s, which pop up in population modelling, and also have interesting mathematical properties. (Note: These are different (though related to) the *logistic curve*, which also pops up in population modelling problems and in calculus classes. If you've heard of that, you can think of these logistic maps as a discrete-time version of that continuous-time population model, but this discrete version has much more personality.)

2.1 Population Modelling

Let's model the size of a population. Suppose there is a species of rabbit that has a fixed generation length. If the rabbits have an infinite supply of food, then after each generation, each rabbit is replaced with r rabbits of the next generation.

Problem 5.

- Given this infinite supply of food, if we start with x rabbits at generation 0, how many rabbits do we have at generation t?
- Under what circumstances will the number of rabbits approach a constant population?

Now assume that the rabbit's reproductive rate depends on the amount of available food, and that the amount of available food depends on the number of rabbits. Assume that their environment has a carrying capacity, a number of rabbits that would be able to eat all the available food in a given year. Let's measure the population not as a natural number, counting the rabbits, but as a real number, x, which is the fraction of the carrying capacity, so that $x = 0$ indicates 0 rabbits, but $x = 1$ indicates that the population is the carrying capacity. Now assume that with each successive generation, each rabbit is replaced with $r(1 - x)$ children, so that as the number of rabbits increases to the carrying capacity, and the amount of available food decreases to 0, the reproductive rate shrinks down from r to 0.

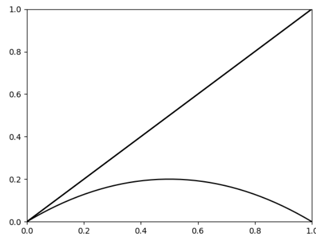
Definition 2.1. If there are x rabbits at generation t, then there will be $rx(1-x)$ rabbits at generation $t+1$. We call this function the *logistic map with parameter* r, and will use the notation

$$\boxed{f_r(x) = rx(1 - x)}$$

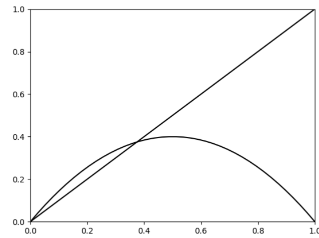
Problem 6.

- Describe the trends in population if $0 \le r < 1$.
- Explain what happens in our model if we let x at generation t be greater than 1 - that is, if we end up with more rabbits than the carrying capacity.
- What population at generation 0 maximizes the population at generation 1?
- We know that our model isn't necessarily predictive if we ever have x outside the interval $[0, 1]$. What values of r guarantee that if we start with $x \in [0, 1]$, it stays in that interval forever?

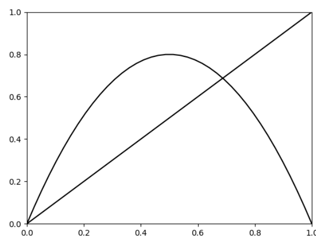
Problem 7. Draw a cobweb diagram for f_r at $r = 0.8, 1.6, 3.2, 3.5$, with one or more different starting values of $0 < x < 1$.



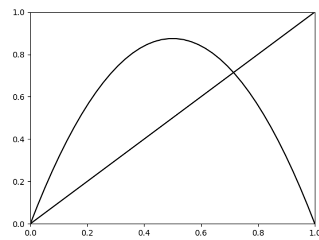
(a) $r = 0.8$



(b) $r = 1.6$



(c) $r = 3.2$



(d) $r = 3.5$

- Of these different values of r, for which does the population approach a stable fixed point?
- If the population doesn't approach a stable fixed point, what happens instead?

Problem 8. What are the fixed points of f_r? Are they valid populations? If f_r has a fixed point, when is it stable?

Problem 9. For what values of $0 \leq r \leq 4$ is 0 a stable fixed point of f_r? For what values is it an unstable fixed point? (You don't need to prove this fully rigorously yet, but give some explanation.)

Problem 10. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant polynomial.

- Show that the fixed points are exactly the zeroes of $f(x) - x$.
- **Optional:** Assume that $f(x)$ is not the polynomial x. Show that there are only countably many values of x such that $x, f(x), f(f(x)), \ldots$ ever exactly reaches a fixed point of f.
- Recall that the finitely many zeroes of f split $\mathbb{R}$ into intervals on which f is positive and intervals on which f is negative. Show that for a fixed point x_0 to be stable, $f(x) - x$ must be positive immediately to the left of x_0 or negative immediately to the right of x_0.

Problem 11. Prove that for $0 \leq r \leq 4$, 0 is a stable fixed point exactly when the previous problem says it should be (with the possible exception of $r = 1$ when something a little bit weird is happening - explain why!). Use your data to conjecture whether this is true for the other fixed point of f_r.

We will find the following theorem about limits useful:

Theorem 2.1. Let $a_0, a_1, a_2, \dots$ be a sequence of numbers, let $0 \leq r < 1$, and let a_∞ be a number. Show that if for all n, $|a_{n+1} - a_\infty| < r|a_n - a_\infty|$, then $a_0, a_1, a_2, \dots$ converges to a_∞.

Problem 12. Prove Theorem 2.1.

Problem 13. Let $f(x) = ax^2 + bx + c$ be a quadratic, and let x_0 be a fixed point of $f(x)$. Show that if $|2ax_0 + b| < 1$, then x_0 is stable.

(Hint: Use Theorem 2.1)

Problem 14. Let $f(x) = ax^2 + bx + c$ be a quadratic, and let x_0 be a fixed point of $f(x)$. Show that if $|2ax_0 + b| > 1$, then x_0 is unstable.

Problem 15. For what values of $0 \leq r \leq 4$ does $f_r(x)$ have a stable fixed point? Does this agree with your observations?

3 Periodic Points and Chaos

If $n > 0$ is a natural number, $f : \mathbb{R} \to \mathbb{R}$ is a function. Then we say that $x \in \mathbb{R}$ is a *periodic point* of f when $f^n(x) = \underbrace{f(f(\dots(f}_{n \text{ times}}(x))\dots) = x$. We say that x *has period* n if n is the least positive number such that this is true.

For instance, under the map $f(x) = -x$, every point is periodic, and every point has period 2 except for 0 which has period 1. A point of period 1 is the same as a fixed point.

Problem 16. From your cobweb plot data on the logistic map, what are some values of r that have periodic points that are not fixed points? What periods do they have? Conjecture what periods are possible for periodic points of f_r with $0 \leq r \leq 4$. You may appreciate the following spreadsheet:

`https://docs.google.com/spreadsheets/d/1dut5fZqMHgsUlmykaylETnBG7hBYPsDSf1Npw_G1GWE/edit?usp=sharing`

Problem 17. Show that if f_r has a point of period 2, then it has 2 points of period 2, and they sum to $1 + \frac{1}{r}$.

Problem 18. Check out the following gif, which animates the behavior of logistic maps over time. Does this match your previous answers as to which values of r had fixed points, or points of a certain period?

`https://en.wikipedia.org/wiki/Logistic_map#/media/File:Logistic_Map_Animation.gif`

3.1 Intermediate Value Theorem and Chaos[1]

Recall the Intermediate Value Theorem:

Theorem 3.1 (**Intermediate Value Theorem**)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on an interval $[a, b] \subset \mathbb{R}$. Moreover, suppose that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = 0$.

Let f be a continuous function such that if $x \in [0, 1]$, then $f(x) \in [0, 1]$, such as f_r for $0 \leq r \leq 4$.

Problem 19. Using the intermediate value theorem or Brouwer's Fixed Point Theorem, show that f has a fixed point in $[0, 1]$.

Definition 3.1. If f has a periodic point of period n in $[0, 1]$ for all n, we call f *chaotic*.

We've seen logistic maps f_r with points of periods $1, 2, 4$, and maybe others, but something special happens when there is a point of period 3:

Theorem 3.2. If f has a point $x \in [0, 1]$ of period 3, then f is chaotic.

To prove this, we will want the following notation:

Definition 3.2.
- Let I be an interval. Then let $f(I)$ be the set of all points $f(x)$ where $x \in I$.
- Let I, J be closed intervals. Say that $I \to J$ whenever $f(I) \supseteq J$.

This theorem is the tool we will use to prove Theorem 3.2. It lets us find periodic points x_0 such that the points in the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ follows a predictable path:

Theorem 3.3 (Itinerary Lemma)**.** Let $I_0, \ldots, I_{n-1}$ be closed intervals, and assume that $I_0 \to I_1 \to \ldots I_{n-1} \to I_0$. Then there is a point $x_0 \in I_0$ such that $f^n(x_0) = x_0$ and for all $k = 0, \ldots, n-1$, $f^k(x_0) \in I_k$.

We will prove this theorem later, using the intermediate value theorem.

Problem 20. Say we find intervals $I_0 \to I_1 \to \cdots \to I_{n-1} \to I_0$ such that for all $k = 1, \ldots, n-1$, I_k does not intersect I_0. Show that f has a periodic point $x_0 \in I_0$ of period n exactly. (You may use the Itinerary Lemma.)

Problem 21. Let $3 < n$, and assume that f has a point with period 3 in $[0, 1]$. We will show that f has a periodic point of period n.

[1] For more about this, check out Padraic Bartlett's notes from Mathcamp 2014: `https://web.math.ucsb.edu/~padraic/mathcamp_2014/mathcamp_2014.html`

- Show that there exist points $0 \leq x_0 < x_1 < x_2 \leq 1$ such that either $f(x_0) = x_1$, $f(x_1) = x_2$, and $f(x_2) = x_0$, or $f(x_0) = x_2$, $f(x_2) = x_1$, and $f(x_1) = x_0$. For the rest of the proof, we will assume that $f(x_0) = x_1$, $f(x_1) = x_2$, and $f(x_2) = x_0$.
- Let $I = [x_0, x_1]$ and $J = [x_1, x_2]$. Show (with the intermediate value theorem) that $I \to J$, $J \to I$, and $J \to J$.
- Conclude that if we let $I_0 = I$, and for $k = 1, \ldots, n-1$, let $I_k = J$, then $I_0 \to I_1 \to \cdots \to I_{n-1} \to I_0$. Using the Itinerary Lemma, prove that f has a point of period n in I.
- Explain how to make this proof work if instead of $f(x_0) = x_1$, $f(x_1) = x_2$, and $f(x_2) = x_0$, we have $f(x_0) = x_2$, $f(x_2) = x_1$, and $f(x_1) = x_0$.

Problem 22. Assume that f has a point of period 3 in $[0, 1]$. Show it has a point of period 2.

All that remains is to prove the Itinerary Lemma!

Problem 23. Use the intermediate value theorem (and a graph, if you'd like) to show that if $I \to J$, then there is some closed subinterval $I^* \subseteq J$ such that $f(I) = J$.

Problem 24. Prove the Itinerary Lemma. Let $I_0, \ldots, I_{n-1}$ be our closed intervals such that $I_0 \to I_1 \to \ldots I_{n-1} \to I_0$.

- Find an interval $I^*_{n-1} \subseteq I_{n-1}$ such that $f(I_{n-1}) = I_0$.
- Find an interval I^*_k for each $k = 0, \ldots, n-2$ such that for each k, $I^*_k \subseteq I_k$, and also $f(I^*_0) = I^*_1, f(I^*_1) = I^*_2, \ldots, f(I^*_{n-2}) = I^*_{n-1}$.
- Prove that $f^n(I^*_0) = I_0$.
- Show that f^n has a fixed point x_0 in I^*_0.
- Show that for all $k = 0, \ldots, n-1$, $f^k(x_0) \in I_k$.

4 Complex Numbers and the Mandelbrot Set

4.1 Complex Review

Definition 4.1.
- Recall that the complex numbers $\mathbb{C}$ can all be expressed as $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
- If $a, b \in \mathbb{R}$, let $\overline{a + bi} = a - bi$. We call this the *complex conjugate* of $a + bi$.
- If $a, b \in \mathbb{R}$, let $|a + bi| = \sqrt{a^2 + b^2}$. We call this the *magnitude* of $a + bi$.

Problem 25. Check that $|a+bi|^2 = (a+bi)(\overline{a + bi})$, and that $|(a+bi)(c+di)| = |a + bi||c + di|$.

Theorem 4.1. The function $d(x, y) = |x - y|$ is a metric on $\mathbb{C}$. In particular, the triangle inequality holds, so if $x, y, z \in \mathbb{C}$, then $|x - y| + |y - z| \geq |x - z|$.

4.2 Bounded Sequences

Definition 4.2. Let $z_0, z_1, z_2, \ldots$ be a sequence of complex numbers. Say that this sequence is *bounded* if and only if there is some real number M such that for all $n \in \mathbb{N}$, $|z_n| \leq M$.

Problem 26. Determine which of the following sequences of complex numbers are bounded:

- $0, 1, 2, 3, \ldots$
- $1, \frac{1}{2}, \frac{1}{3}, \ldots$
- $z_0, z_1, z_2, \ldots$ where $z_n = \cos(n) + i\sin(n)$.
- $\frac{1}{1^2}, \frac{1}{1^2} + \frac{1}{2^2}, \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}, \ldots$
- $\frac{1}{1}, \frac{1}{1} + \frac{1}{2}, \frac{1}{1} + \frac{1}{2} + \frac{1}{3}, \ldots$

Problem 27. Let $c \in \mathbb{R}$ be positive, and let $z_0, z_1, \ldots$ be a sequence of complex numbers. Assume that $|z_0| > c$, and that there is some real number d such that for all n, $|z_{n+1}| - c > (|z_n| - c) + d$.

- Prove by induction that for all n, $|z_n| \geq c + dn$.
- Prove that $z_0, z_1, z_2, \ldots$ is unbounded.

Problem 28. Let $c \in \mathbb{R}$ be positive, and let $z_0, z_1, \ldots$ be a sequence of complex numbers. Assume that $|z_0| > c$, and for all n, $|z_{n+1}| - c \geq 2(|z_n| - c)$.

- Prove by induction that for all n, $|z_n| \geq c + 2^n(|z_0| - c)$.
- Prove that $z_0, z_1, z_2, \ldots$ is unbounded.

4.3 Basics of the Mandelbrot Set

Let c be a complex number. Define the function $f_c : \mathbb{C} \to \mathbb{C}$ by $f_c(z) = z^2 + c$.

Problem 29. Define the sequence $z_0(c), z_1(c), z_2(c), \ldots$ by letting $z_0(c) = 0$, and then once $z_n(c)$ is defined, define $z_{n+1}(c) = f_c(z_n(c))$.

Determine whether the sequence $z_0(c), z_1(c), z_2(c), \ldots$ is bounded for the following choices of c:

$$c = 0, 1, -1, i, -i, \frac{1}{4}$$

Definition 4.3. We define the *Mandelbrot set* to be the set of all complex numbers c such that the sequence $z_0(c), z_1(c), \ldots$ is bounded.

Problem 30. We will show that the Mandelbrot set is contained in the disk of radius 2 centered at the origin.

- Let $c, z \in \mathbb{C}$ be such that $|z| > 2$. Then show that $|f_c(z)| - |c| \geq 2(|z| - |c|)$.

- Show that if for some n, $|z_n(c)| > 2$ and $|z_n(c)| > |c|$, then the sequence $z_n(c), z_{n+1}(c), z_{n+2}(c), \ldots$ is unbounded.
- Conclude that if for some n, $|z_n(c)| > 2$ and $|z_n(c)| > |c|$, then c is not in the Mandelbrot set.
- Finally, show that if $|c| > 2$, then c is not in the Mandelbrot set, and conclude that the Mandelbrot set is contained in the disk of radius 2 centered at the origin.

Problem 31. Show that c is in the Mandelbrot set if and only if $|z_n(c)| \leq 2$ for all n.

Problem 32. Show that if c is in the Mandelbrot set, then $\bar{c}$ is in the Mandelbrot set. Conclude that the Mandelbrot set is symmetric around the real line.

4.4 Mandelbrot set and Logistic Maps

The Mandelbrot set has a deep connection with logistic maps, that will allow us to understand the intersection of the Mandelbrot set with the real line.

Problem 33. Let r be a real number.

- Find real numbers a, b, and c such that $(ax+b)^2 + c = af_r(x) + b$.
- Assume $r \neq 0$. Conclude that for $x_0 \in \mathbb{R}$, x_0 is a periodic point of f_r if and only if $ax_0 + b$ is a periodic point of $x^2 + c$.
- Using the connection to the logistic map, prove that the interval $[-2, 1/4]$ on the real line is contained in the Mandelbrot set.

Problem 34. (Challenge) Stable fixed points can be defined for functions $f : \mathbb{C} \to \mathbb{C}$, in essentially the same way they were defined over the reals.

- Show that if r is complex, then $f_r(z) = rz(1-z)$ defined on the complex numbers has a stable fixed point whenever $|r| < 1$.
- Show that if $|r| < 1$ and $c = \frac{r}{2}\left(1 - \frac{r}{2}\right)$, then the map $z^2 + c$ has a stable fixed point. In fact, c is in the Mandelbrot set.
- Any complex r with $|r| = 1$ can be expressed as $\cos\theta + i\sin\theta$ for some real θ. Find a parametric equation for the set of all $c \in \mathbb{C}$ such that $c = \frac{r}{2}\left(1 - \frac{r}{2}\right)$ for some r with $|r| = 1$. Graph this curve, and use it to identify the part of the Mandelbrot set that consists of c such that you know (based on part (a)) that $z^2 + c$ has a stable fixed point. (In fact, these will be all the values of c such that $z^2 + c$ has a stable fixed point.)

5 Video

To find out more about these topics, Youtube has many great videos about the logistic map, the Mandelbrot set, and the connections between the two.

Here is one of my favorites, on the logistic map, which touches on the Mandelbrot set: `https://www.youtube.com/watch?v=ovJcsL7vyrk`