Surreal Numbers

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1 Historical Note: Party Parrots

You may be able to deduce from the date on this pdf that this class was taught online, at a camp mostly run on Slack, where party parrot emojis were a recurring meme. At the Relay competitions this camp, there were lots of games played by moving parrots around on Google Jamboards, and thus that's what we did this camp. I'm including the jamboard links in this pdf, but in case they're broken, I've also saved them as accompanying pdfs.

Here's the Impartial Games Jamboard and the Partizan Games Jamboard.

2 Impartial Games

Definition 1. A *combinatorial game* is a game that is

- two-player (two players alternate turns)
- complete-information (no face-down cards)
- deterministic (no randomness)
- well-founded (it's guaranteed to finish in a finite amount of turns)
- where the first player who cannot move loses, and thus the other player wins

We will start by studying *impartial* combinatorial games, which are games where the rules are the same for each player. It may still be better to be the player who goes first or second.

Problem 1. Here are some familiar examples of games. Do these meet the definitions, and are they impartial?

- Chess
- Checkers
- Poker
- Football (whatever that means to you)

When we study games, we will think about *positions* - a position is the state of the game, recording the possible moves for each player, right before a particular turn. In each impartial combinatorial game, each position is either a guaranteed win for the Next player to move, and known as an *N-position*, or a guaranteed win for the Previous player, a *P-position*. For instance, if there are no legal moves from a given position, it is a P-position, because the next player is out of luck and the previous player to move just won.

*with thanks to John Conway and Alfonso Gracia-Saz

2.1 Nim

Perhaps the most classic impartial game to study is Nim. If you've been participating in relays this summer - the format will be pretty familiar. There are a few piles of objects (for our purposes, a few different groups of party parrots on a Jamboard, but I recommend beans or coins if you play this IRL). On each turn, you take a positive number of parrots/beans/coins from *exactly one* pile. Play proceeds until someone empties the last pile, and thus wins, because there is no valid move for the other player.

Problem 2. I've prepared a jamboard with some Nim games, have fun! If your group has some people who have not played Nim before, they should try to play each other first, and then face more experienced players.

Play Impartial Games Jamboard slides 1 - 3.

Problem 3. Say we're playing Nim with two piles - with m and n parrots respectively (m and n could be zero, in which case there are really 0 or 1 piles). For what values of m and n is this a P-position, or an N-position?

Problem 4. Say you have an impartial game, and an algorithm for determining whether any given position is a P- or N-position. Describe a winning strategy for the game, assuming you get to choose whether you go first or second at the beginning.

2.2 Other Subtraction Games

Subtraction games are variations on Nim - still impartial games. Let S be a set of positive integers. Then once again let us populate some piles with parrots, and once again you remove them from one pile at a time, but now you are only allowed to move a number of parrots in S.

Problem 5. Try Impartial Games Jamboard slides 4 through 6 - each will specify a set S. Feel free to mix and match S with the configuration, and generally play games as much as you want until you get a feel for the strategy.

2.3 Adding Games Together

I want to teach you a new board game. It's called *Chess + Checkers*. To play, we sit down on opposite sides of a table, and set up a chessboard and a checkerboard side-by-side. On each of our turns, we pick one of the boards and do a move in that board. As if chess and checkers aren't hard enough (well, maybe checkers isn't hard enough), now you have to decide whether to respond to my threat in chess, or get ahead in checkers.

In general, we can add any two combinatorial games G_1, G_2. In the new game, $G_1 + G_2$, each player gets to move in only one game at a time, and the person who runs out of moves in *both* games loses. (That is, I could win G_1, but if we're still playing G_2, it doesn't matter unless I can win that too.)

Problem 6. Convince yourself that the sum of two impartial games is impartial, the sum of two games of Nim is a game of Nim (how many piles are there? of what sizes?).

Problem 7. Is this addition commutative? Is it associative?

Problem 8. Let G be an impartial game. Who wins $G + G$?

Definition 2. We say that two games G_1, G_2 are *equivalent* (written $G_1 \approx G_2$) if for every game H, the sum $G_1 + H$ starts in a P-position if and only if the sum $G_2 + H$ does.

Problem 9. Show that this notion of equivalence is an *equivalence relation*:

- For any game G, $G \approx G$
- For any games G_1, G_2, $G_1 \approx G_2$ if and only if $G_2 \approx G_1$
- For any games G_1, G_2, G_3, if $G_1 \approx G_2$ and $G_2 \approx G_3$, then $G_1 \approx G_3$.

Problem 10. Show that if $G_1 \approx G_2$ and $H_1 \approx H_2$, then $G_1 + H_1 \approx G_2 + H_2$.

Definition 3. Let 0 be the game with no valid moves for either player (for instance, Nim with 0 piles).

Problem 11. Show that for any game G, $G + 0 = G$.

Problem 12. Show that a game G is equivalent to 0 if and only if G starts in a P-position.

Problem 13. Let G_1, G_2 be impartial games. Prove that $G_1 \approx G_2$ if and only if $G_1 + G_2 \approx 0$.

2.4 Nimbers

Definition 4. Let G be an impartial game with only finitely many valid moves at each position (like, for instance, Nim with finitely many parrots).

Define the *Sprague-Grundy* value of G as follows:

First calculate the Sprague-Grundy value of each position you can legally move to. Then let the Sprague-Grundy value be the smallest natural number that isn't on that (finite) list.

Problem 14. For a natural number n, let $*n$ represent a Nim game with one pile of n parrots (Here $*0 = 0$). What is the Sprague-Grundy value of $*n$?

Problem 15. Prove that any impartial game with Sprague-Grundy value n is equivalent to $*n$. (Hint: Assume this is true for every position you can move to, and do a kind of induction.)

This is actually pretty remarkable. Any finite-move impartial game is equivalent to a single pile of Nim! This even gives us a winning strategy! For finite-move impartial games, we can recursively determine any position's Sprague-Grundy value, and then we know whether that position is a P-position or an N-position! Then it's just a matter of trying to move to P-positions whenever possible.

Problem 16. Let's give a problem from Friday's relays another shot. There is one pile of n parrots, and at each turn, you must remove a number of parrots which is a power of 4. What is the Sprague-Grundy value of this game for each n? For which values of n is this a P- or N-position?

Problem 17. Let's go back to two-pile Nim, which we now refer to as $*m + *n$. What's the Sprague-Grundy value? (Hint: You will want to write m and n in binary.)

Using this, describe a strategy for general Nim, that is, $*n_1 + *n_2 + \cdots + *n_k$.

We've now developed the basic theory of nimbers. A *nimber* is just an impartial game, where two nimbers are considered the same if the games are equivalent. We've proven (at least in the finite-move case) the *Sprague-Grundy Theorem*, which says that each impartial game is equivalent to a single-pile version of Nim, so each nimber can be represented as a single-pile version of Nim, hence the name. These have a special addition, an additive identity, and in fact they follow the axioms of an additive abelian group (remember to replace equality with $\approx$).

Problem 18. For what values of n is the set $\{*0, *1, *2, \ldots, *n\}$ closed under addition? (That is, if $i, j < n$, then $*i + *j = *k$ for some $k < n$.) Bonus: Check that for those values of n, this set is an abelian group. Is this isomorphic to any groups you know?

3 Infinite Nimbers

Now let's see if we can move past the finite-move requirement to infinite Nim, infinite nimbers, and all impartial games.

To do this, we're going to need to work in more set theory. We can think of any impartial game as a set! If G is an impartial game, we can instead think of it as the set of all positions you can move to - where each of those positions is in turn represented by the set of moves we can get to from there, and so on. For instance, we can write $*n$ as $\{*0, *1, \dots, *(n-1)\}$.

For any set of impartial game positions, you can form another impartial game, where that set is the set of valid moves. Note that the original position will never be one of those moves, as that would allow an infinite chain of turns without an end to the game. To ensure that our infinite games are well-founded (do not have infinite chains of turns), we can just make sure that we write them as a set of games that we've already constructed and checked are well-founded. One such game is $*\omega = \{*0, *1, *2, \dots\}$ (ω is pronounced "omega"). What does ω actually look like if we think of it as Nim? It should be just one pile, and it should have infinitely many parrots, but whenever we remove parrots, we should be left with only finitely many. To model this, think of the parrots as piled in an infinite stack, with a parrot at the bottom, but they go allll the way up. Now, a valid Nim-move is to pick a parrot in the stack, and remove that parrot along with all the parrots above it. From this point on, when dealing with infinite Nim-piles, the order will matter, and we'll specify in the rules that you can only take parrots from the top of the pile. (Also, when I say a "pile" or a "stack" of parrots, I'm assuming that it goes straight up - if you had two parrots both sitting on the same other parrot, that'd probably make a "tree".)

Problem 19. Assume we have a well-ordered pile of parrots - that is, every nonempty set of parrots in the pile has a bottommost parrot. Prove that this game of one-pile Nim must end eventually (although it may take a long, long, long time).

Conversely, show that if we have a valid one-pile Nim game, the parrots must be well-ordered.

It is now time to bring in the ordinal numbers. They're an ordered number system that starts with the natural numbers and then blasts off into infinity. Rather than define them explicitly, I'm going to refer to specific facts about them whenever I need to.

Here are some of those facts: the ordinals are well-ordered, and thus every set of ordinals is well-ordered. In particular, for every ordinal number α, the set of ordinals less than α is a well-ordered set. In fact, every well-ordered set is *order-isomorphic* to the set of ordinals less than some unique α. This means that if we have a well-ordered pile of parrots, we could number the parrots in increasing order with the ordinals less than α. We then say that the pile has *height* α. For instance, $*\omega$ as defined before consists of a pile of height ω - where ω is the smallest infinite ordinal. There is one parrot for each number below ω, that is, for each natural number.

The ordinals also have an addition defined on them - for two ordinals α, β, to define $\alpha + \beta$, take a stack of parrots of height α, and put a stack of parrots of height β on top.

Problem 20. This addition is not commutative - why is $\omega + 2$ not equal to $2 + \omega$?

For each ordinal α, we define an infinite nimber $*\alpha$.

Problem 21. Show that for distinct ordinals $\alpha \neq \beta$, $*\alpha$ and $*\beta$ are not equivalent games.

We are now just about ready to define Sprague-Grundy values for arbitrary impartial games.

Problem 22. Define an ordinal-valued Sprague-Grundy value for all impartial games. It should agree with the natural number S-G value for games with only finitely many moves at each position.

Prove that for your definition, $G \approx *\alpha$ if and only if α is the S-G value.

Once we have defined Sprague-Grundy values, and we have a notion of infinite Nim piles, we can use multi-pile Nim to define addition.

Problem 23. If $n \in \mathbb{N}$, then what is $*\omega + *n$?

Problem 24. For which ordinals α is the set $\{*\beta : \beta < \alpha\}$ of nimbers (the set of nimbers "under α") closed under addition? (When this happens, the set is an abelian group.)

Problem 25. If α is an ordinal such that the set of nimbers under it is closed under addition, and $\beta < \alpha$, what is $*\alpha + *\beta$?

4 Multiplying Nimbers

We can even define multiplication of nimbers (this will turn them into a ring, and even a field). It's less game-theoretically clear what multiplication should mean, but there is still a sense in which this is the "right" multiplication.

Problem 26. Prove that if we have a sensible definition of multiplication (satisfying associativity, distributivity with addition, one is an identity, and no zero divisors) then for all $x' \neq x$ and $y' \neq y$, we have $xy \neq x'y + xy' - x'y'$.

Problem 27. Let $n \in \mathbb{N}$. Calculate $*2^{2^n} \cdot *2^{2^n}$. Assume $m < 2^{2^n}$. Calculate $*m \cdot *2^{2^n}$.

Explain how you can then use this, together with binary expansions, to calculate Nim-multiplication in general.

Problem 28. Design a multiplication for nimbers that satisfies the requirement of the last problem. For $n \in \mathbb{N}$, under what circumstances are the nimbers under $*n$ closed under both addition and multiplication? (They then form a ring.)

Problem 29. Another good property for multiplication to follow is having an inverse operation, which we call division. If α is an ordinal, find an impartial game (described as a set S of moves) such that the S-G value of S is $(*\alpha)^{-1}$.

Problem 30. Prove that any $n \in \mathbb{N}$ such that the set of nimbers under $*n$ is closed under addition and multiplication is also closed under inversion (and is thus a field). If you know something about finite fields, which finite fields have we constructed?

Problem 31. For an ordinal α, find an impartial game S with S-G value $\sqrt{*\alpha}$. (Hint: you want to express this as $\{\sqrt{\alpha'} : \alpha' < \alpha\} \cup \{f(\beta, \beta') : \beta, \beta' < \alpha \text{ and } \beta \neq \beta'\}$ for some function f.)

Now for some food for thought. This stuff is a bit too wild for me to write problems out of it, but it should make for an exciting glimpse into the larger world of Nimbers. The general motivating rule for algebraic constructions involving nimbers is that everything is the smallest possible value. The sum $*\alpha + *\beta$ is the smallest that it reasonably could be, the product $*\alpha \cdot *\beta$ is the smallest it reasonably could be, etc. Conversely, if the set of nimbers under $*\alpha$ is not closed under addition, then $*\alpha$ is the sum of the (lexicographically) smallest pair $(*\beta, *\gamma)$ that doesn't have a sum less than α. If the set of nimbers under $*\alpha$ is not closed under multiplication, then α is the product of the smallest pair with no product yet. If the set of nimbers under $*\alpha$ is not closed under inverses, then α is the inverse of the smallest number that doesn't yet have an inverse.

This even applies to polynomials - if the set of nimbers under $*\alpha$ isn't an algebraically closed field, then α is the solution to the lexicographically smallest polynomial that doesn't have a solution. This leads to perhaps the most ridiculous calculation I've ever seen in my life, leading to the amazing theorem: the smallest ordinal α such that the set of nimbers under α is algebraically closed is $\omega^{\omega^{\omega}}$.

5 Partizan Games

It's now time to start thinking about games that aren't impartial. We can't really call them "partial" games, because that sounds like it means a portion of a game. Instead, we call them *partizan* games. This makes some sense because in these games, with different options available to each player, there may be a major bias in one person's favor. Except for the "z" - that doesn't really make sense, but Conway insisted on it.

A general combinatorial game (impartial or partizan) can be modeled as a pair of sets of games, written $\langle G_L | G_R \rangle$, where G_L is the set of moves available to the player on the left, and G_R is the set of moves available to the player on the right. Note that this is different from the first or the second player - you can imagine, perhaps, the two players flipping a coin to decide who moves next.

Problem 32. How can you represent an impartial game as a pair $\langle G_L | G_R \rangle$?

5.1 Partizan Nim

It's time to play games again! Let's play a variation on Nim, where now the two players (left and right) can move different parrots. In addition to the normal parrots we had before (in green), we add devil red parrots and angel blue parrots. It is now important to define the order of the parrots in each pile - you choose a parrot to remove, and remove with it every parrot above it. Adding even more restrictions, only the player on the left can choose to remove a blue parrot with all the parrots above it, and only the player on the right can remove a red parrot with all the parrots above it.

Problem 33. Play the Partizan Games Jamboard. (Play around to get a feel for what these new rules are like, but remember that the rest of the worksheet will outline the general strategy.)

5.2 Measuring Advantage

You may have noticed that having a pile of n red or n blue parrots just gives n free moves to one player or the other. Seeing things from the point of view of the left player, we give some games numerical values, indicating the advantage that the left player has over the right player. For this reason, we call the game with one pile of n blue parrots n, and the game with one pile of n red parrots $-n$.

With Partizan games, we can use the same definition of 0 and addition, which is still commutative and associative and has 0 as an identity for the same reasons. We also have the same definition of equivalence, where $G_1 \approx G_2$ when $G_1 + H$ always has the same outcome as $G_2 + H$. In particular, a game is still equivalent to 0 if and only if the second player always wins.

Problem 34. Show that addition on these "integer" games agrees with addition on actual integers, up to equivalence.

Problem 35. If $G = \langle G_L | G_R \rangle$ and $H = \langle H_L | H_R \rangle$, then find an expression for the left and right move sets of $G + H$.

Problem 36. Find a definition of the *negative* of a game such that $G + (-G) \approx 0$. (Hint: the negative of an impartial game will be itself!)

Problem 37. Consider a game with one red parrot sitting on one blue parrot. If this also represents a number, what should that number be?

5.3 Ordering

Let G be a (potentially partizan) game. There are actually four possibilities for the outcome of G, and we have only explored two. Either the first player always wins, the second player always wins, the left player always wins, or the right player always wins. We will try to define an ordering on games that encodes these possibilities - if the second player always wins, then $G \approx 0$. If the left player always wins, then we call G *positive*, and say $G > 0$. If the right player always wins, we call G *negative*, and say $G < 0$, and if the first player always wins, then we call the game *fuzzy*, denoted $G||0$. To make the ordering do this, we want $G \leq H$ to mean that somehow H is more biased towards the left player than G is, no matter who goes first. Here's the actual definition that does this:

Definition 5. Let $G = \langle G_L|G_R\rangle$ and $H = \langle H_L|H_R\rangle$ be two games. We can define an ordering as follows: $G \leq H$ when there is no $h_R \in H_R$ with $h_R \leq G$, and no $g_L \in G_L$ with $H \leq g_L$.

The idea here is that every move the right player can make from H is worse for the right player than G, and every move the left player can make from G is worse for them than just playing H.

This, it turns out, is very nearly a partial ordering - it is reflexive (for all G, $G \leq G$) and it is transitive (for all G, H, K, if $G \leq H$ and $H \leq K$, then $G \leq K$). However, it is not quite antisymmetric - if $G \leq H$ and $H \leq G$, we don't quite have $G = H$ - but this happens exactly when $G \approx H$, as we will prove later. Thus if we consider games only up to equivalence (as we will), this ordering actually is antisymmetric.

Problem 38. Show that $\leq$ is reflexive and transitive.

Problem 39. Prove that $\leq$ actually does what we want it to - that is, if the left player moves second, then they win iff $0 \leq G$, and if the right player moves second, then they win iff $G \leq 0$.

Problem 40. Let $G = \langle G_L|G_R\rangle$, and let $g_1, g_2 \in G_L$. Assume that $g_1 \leq g_2$. Then show that G is equivalent to the game we get by removing g_2. A move we can just remove like that is called a *dominated* move.

Problem 41. Let $G = \langle G_L|G_R\rangle$, and let $g_1 \in G_L$ and let $g_2 \in (g_1)_R$, that is, a right-move of g_1. Assume that $g_2 \leq G$. Show that G is equivalent to the game we get when we substitute g_1 with all the moves of $(g_2)_L$. A move like g_1 is called *reversible*, and if we replace g_1 with $(g_2)_L$, we have *bypassed* g_1.

Problem 42. Simplify these games as much as you can, by using the last two problems:

- $G_1 = \{3, 5| - 2\}$
- $G_2 = \{3, 3 + \frac{1}{2}| - 2\}$
- $G_3 = \{3, 2| - 2\}$
- $G_4 = \{1, -1|\}$
- $G_5 = \{0, 1, -1|0, 1, -1\}$

5.4 More Addition

Problem 43. Show that adding an impartial game to a positive game gives a positive game, and adding an impartial game to a negative game gives a negative game. (No amount of impartiality can outweigh even a small partizan advantage.)

Problem 44. Show that if $G_1 \leq G_2$, then $G_1 + H \leq G_2 + H$. Use this to show that if $G_1 \approx G_2$ and $H_1 \approx H_2$, then $G_1 + G_2 \approx H_1 + H_2$.

Problem 45. Describe how you can find out whether $G \leq H$ and whether $H \leq G$ by looking at the outcome of $G - H$.

Problem 46. Show that $G_1 \leq G_2$ and $G_1 \leq G_2$ if and only if $G_1 - G_2 \approx 0$ if and only if $G_1 + H \approx G_2 + H$.

6 Surreal Numbers

We've introduced some surreal numbers already - namely the integer surreals. Now we can give a definition:

Definition 6. A surreal number is a game $\langle G_L | G_R \rangle$ (mod equivalence) where G_L and G_R are sets of surreal numbers such that there is no pair $g_l \in G_L$ and $g_r \in G_R$ such that $g_r \leq g_l$. We'll probably just call them *numbers* from here on out. (A game is still a number if it's just equivalent to a number.)

Problem 47. Check that the numbers defined so far are, in fact, numbers, and that no impartial game is. (Sorry, nimbers.)

Problem 48. Show that for any two numbers G, H, either $G \leq H$ or $H \leq G$ (or both, if $G \approx H$.)

Problem 49. Show that numbers are closed under addition.

6.1 Not-Quite-Numbers

Let's take a quick look at how numbers interact with other games, like nimbers.

Problem 50. If x is a number, find a simplified expression for the left and right moves of $x + *1$.

Problem 51. Let $\uparrow = \langle 0 | *1 \rangle$, and $\downarrow = \langle *1 | 0 \rangle$. Show (by structural induction) that for any positive number x, $-x < \downarrow < 0 < \uparrow < x$.

Problem 52. Find a partizan Nim game equivalent to $*1 + \uparrow$.

6.2 Birthdays and the Simplicity Rule

Numbers are built recursively out of other numbers, so we can even play out a process of creating them in sequence. On each step (or "day"), we create all the numbers whose moves are numbers we've already created.

On Day 0, we create all the numbers with moves we've already created - which is to say, just $\langle | \rangle = 0$, as there are no already-created numbers to move to.

Problem 53. What games are created on Day 1? (We already have names for these games.) How many does this bring us to, up to equivalence?

Problem 54. What games are created on Day 2? How many does this bring us up to, up to equivalence? (We already have names for all of these, too!)

Problem 55. How many games should we expect on Day 3? Check your guess. We haven't come up with names for all of these yet, but you can probably come up with some guesses as to what familiar rational numbers they are. Optional: check

The following lemma is super-useful for understanding games created at finite days. You absolutely have the tools to prove it, but you can just assume it, and come back to prove it later.

Lemma 6.1. *Every game created at Day n is either of the form $\langle | G \rangle$, where G is the smallest game so far, or $\langle G | H \rangle$, where there are no games so far between G and H, or $\langle H | \rangle$, where H is the largest game so far.*

Problem 56. How many games are created through Day n?

Problem 57. What is the numerical value of the largest and smallest games created on Day n? (We already have names for these numbers.)

Problem 58. If G, H are games created by Day n, then $\langle G|H\rangle + \langle G|H\rangle \approx G+H$. Use this principle to determine the numerical values of the games created on Day 3.

In order to get all the surreal numbers we might possibly want, we need to play out this process for infinite durations. On Day ω, we create all the numbers whose moves are numbers created on the finite days, and so on for other ordinals.

Problem 59. Which numbers are created on the finite days? Check that this set is closed under addition.

Problem 60. Every real number that hasn't been created on a finite day is created on Day ω. Assuming this, find an expression for $\frac{1}{3}$, either as a pair of sets of moves, or as a pile of parrots. Explain how you might do this for other real numbers.

On Day ω, we also create numbers that we will call $-\omega, -\frac{1}{\omega}, \frac{1}{\omega}$, and ω.

Problem 61. Find expressions for $-\omega, -\frac{1}{\omega}, \frac{1}{\omega}$, and ω that use only numbers created on finite days.

Problem 62. Explain why we should think of the number $\langle 0, 1, 2, \ldots |\omega\rangle$, created on Day $\omega + 1$, as $\omega - 1$.

The manipulations so far can be difficult, but the following theorem lets us more easily recognize numbers we create:

Theorem 6.1 (The Simplicity Rule). *If $G = \langle G_L|G_R\rangle$ is a number, then in fact it is the* simplest *number such that for all $g_l \in G_L$ and for all $g_R \in G_R$, $g_l < G < g_R$. By the* simplest*, we mean created first.*

This means that as long as we understand how numbers are created on Day α, for each ordinal α, we can determine the value of a number without resorting to algebra.

6.3 Multiplication

We can define multiplication on surreal numbers similarly to how it was defined for nimbers: find basic properties we want multiplication to satisfy, and let the product of two numbers be the simplest thing it can be under those constraints.

Problem 63. Show the following in an ordered field (a number system with commutative addition, subtraction, commutative multiplication, division, and an ordering, that all play well together): for all $x_1 < x_2, y_1 < y_2$, we have $x_2y_2 > x_1y_2 + x_2y_1 - x_1y_1$. Using symmetry, find 3 other similar inequalities.

Problem 64. Based on the previous problem and the simplicity rule, define the product of two numbers.

Problem 65. Using the simplicity rule, show that the product of two real numbers (created on/by Day ω) is actually what it should be. This should justify our having called numbers things like $\frac{1}{3}$, as you will actually get $3 \cdot \frac{1}{3} = 1$.

In fact, this addition and multiplication makes the wide class of surreal numbers into an ordered field, a massive one including all of the ordinals. It also has the interesting property that every positive number has a square root, and every odd-degree polynomial has a root - making it a *real closed field*. This is part of the proof of the extra-remarkable fact that people from my model theory class last week will understand - that the surreals are an *elementary extension* of the reals, in the language $\{0, 1, +, *, \leq\}$.