Bounds on the Sizes of Distal Cell Decompositions

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VC-Density

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\varphi(x;y)$ an $\mathcal{L}$-formula, where x and y may be tuples of variables, with length $|x|$ and $|y|$.

Definition

- Define $\pi_\varphi(n)$ to be the maximum over all $A \subset M^{|x|}$ with $|A| = n$ of the number of subsets of A that can be written as $\varphi(M^{|x|}, b) \cap A$ for some $b \in M^{|y|}$.
- Define $\mathrm{vc}(\varphi)$ to be the infimum of all reals d such that $\pi_\varphi(n) = \mathcal{O}(n^d)$, or ∞ if none exist.

Dual VC-Density

Now let $\Phi(x; y)$ be a finite set of $\mathcal{L}$-formulas all of the form $\varphi(x; y)$, with fixed length $|x|$ and $|y|$.

Definition

- Define $S^{\Phi}(S)$ to be the set of complete Φ-types over a set $S \subseteq M^{|y|}$ of parameters, or alternately, the set of maximal consistent subsets of $\{\varphi(x; b) : \varphi \in \Phi, b \in S\} \cup \{\neg\varphi(x; b) : \varphi \in \Phi, b \in S\}$.
- Let $\pi^*_{\Phi}(n)$ be the maximum over all $S \subset M^{|y|}$ with $|S| = n$ of $\left|S^{\Phi}(S)\right|$.
- Let $\mathrm{vc}^*(\Phi)$ be the infimum of all d such that $\pi^*_{\Phi}(n) = \mathcal{O}(n^d)$, or ∞ if none exist.

Dual VC-Density Example

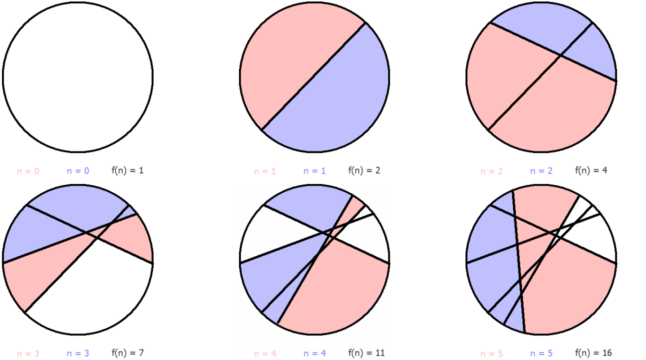

Figure: The maximum number of pieces that n slices cut a circle into[3]

Dual VC-Density Example

- Let $\mathcal{M} = \langle \mathbb{R}; 0, 1, +, -, \cdot, \leq \rangle$, and let $\varphi(x; y)$ be $y_1 x_1 + y_2 x_2 \leq 1$, with $\Phi = \{\varphi\}$.
- For each $b \in M^2$, $\varphi(M; b)$ is a half-plane.
- $\pi^*_\Phi(n)$ is the maximum number of pieces that n half-planes cut $\mathbb{R}^2$ into.
- By induction, $\pi^*_\Phi(n) = \binom{n}{2} + \binom{n}{1} + \binom{n}{0} = \mathcal{O}(n^2)$, and $\mathrm{vc}^*(\Phi) = 2$.

VC-Density and NIP

Fact

If $\varphi^*(y;x) \iff \varphi(x;y)$, *then* $\mathrm{vc}^*(\{\varphi\}) = \mathrm{vc}(\varphi^*)$*.*

Fact

$\mathrm{vc}(\varphi) = \infty \iff \mathrm{vc}^*(\varphi) = \infty$*. In this case, we call* φ *independent.*

Definition

$\mathcal{M}$ is *NIP* (has the *Non-Independence Property*) when every $\varphi(x;y)$ has $\mathrm{vc}(\varphi) < \infty$.

NIP Structures

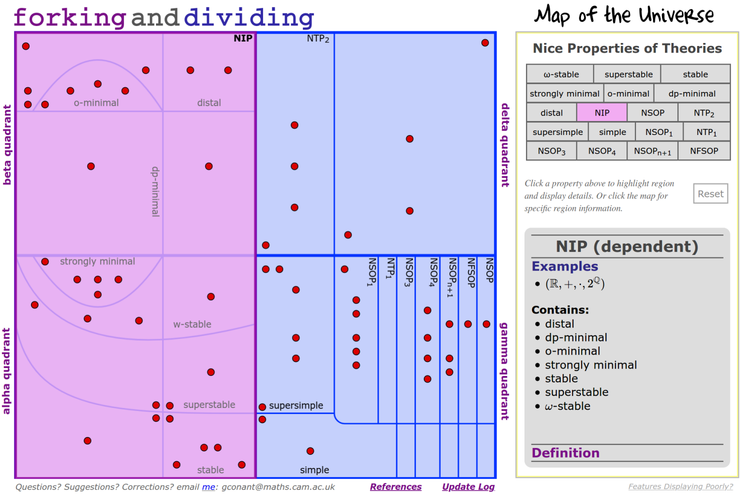

Figure: [4]

Abstract Cell Decompositions

Definition

- If A, X are sets, then X *is crossed by* A if $A \cap X \neq \emptyset$ and $X \setminus A \neq \emptyset$.
- If $X \subset M^{|x|}$, $S \subset M^{|y|}$, then X is crossed by $\Phi(x; S)$ if for some $\varphi \in \Phi$ and some $b \in S$, X is crossed by $\varphi(M; b)$.

Definition

An *abstract cell decomposition* for $\Phi(x; y)$ is a function $\mathcal{T}$ that assigns to each finite $S \subset M^{|y|}$ a set $\mathcal{T}(S)$ whose elements, called cells, are not crossed by $\Phi(x; S)$, and cover $M^{|x|}$ so that $M^{|x|} = \bigcup \mathcal{T}$.

Abstract Cell Decomposition Example

Definition (Dual VC Cell Decomposition)

- Given $\Phi(x; y)$ and a finite $S \subset M^{|y|}$, let the cells of $\mathcal{T}_{\mathrm{vc}}(S)$ correspond to the types in $S^{\Phi}(S)$:
- For each type $p(x) \in S^{\Phi}(S)$, take $\{x \in M^{|x|} : \mathcal{M} \models p(x)\}$ as a cell.

Fact

- *$\mathcal{T}_{\mathrm{vc}}$ is an abstract cell decomposition for Φ*
- *If $\mathcal{T}$ is an abstract cell decomposition for Φ, then for all S, $|\mathcal{T}(S)| \geq |\mathcal{T}_{\mathrm{vc}}(S)|$.*

Abstract Cell Decomposition Example

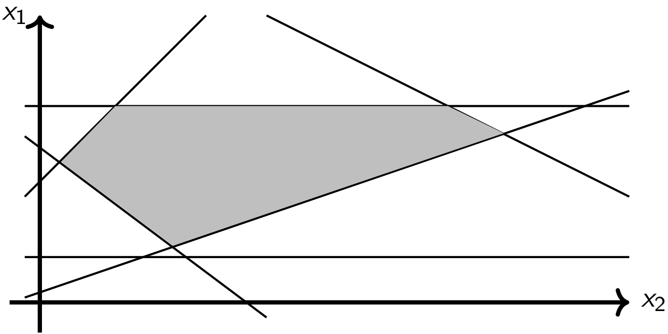

Figure: A cell of $\mathcal{T}_{\text{vc}}(S)$, where $\Phi(x; y) = \{x_1y_1 + x_2y_2 \leq 1\}$, and $\mathcal{M} = \langle \mathbb{R}; 0, 1, +, -, \cdot, \leq \rangle$.[1]

Distal Cell Decompositions

Definition

A *distal* cell decomposition for $\Phi(x; y)$ is an abstract cell decomposition $\mathcal{T}$ whose cells are *uniformly definable* in the following way:

- Each cell is defined by an instance of one of a finite set $\Psi(x; y_1, \ldots, y_k)$ of formulas where $|y_1| = \cdots = |y_k| = |y|$.
- For a given S, the set of potential cells is $\Psi(S) := \{\psi(M^{|x|}; b_1, \ldots, b_k) : \psi \in \Psi, b_1, \ldots, b_k \in S\}$
- For each $\psi \in \Psi$, there is a formula $\theta_\psi(y; y_1, \ldots, y_k)$.
- For each potential cell $\Delta = \psi(M^{|x|}; b_1, \ldots, b_k)$, let $\mathcal{I}(\Delta) = \theta_\psi(M^{|y|}; b_1, \ldots, b_k)$, then exclude Δ if $\mathcal{I}(\Delta) \cap S \neq \emptyset$.
- $\mathcal{T}(S) = \{\Delta \in \Psi(S) : S \cap \mathcal{I}(\Delta) = \emptyset\}$

Distal Structures

Theorem (Chernikov, Galvin, Starchenko)

A structure $\mathcal{M}$ is distal if and only if every finite set of formulas $\Phi(x; y)$ admits a distal cell decomposition.

Distal Cell Decompositions can be thought of as a different perspective on Strong Honest Definitions, a characterization of distality.

Distal Structures

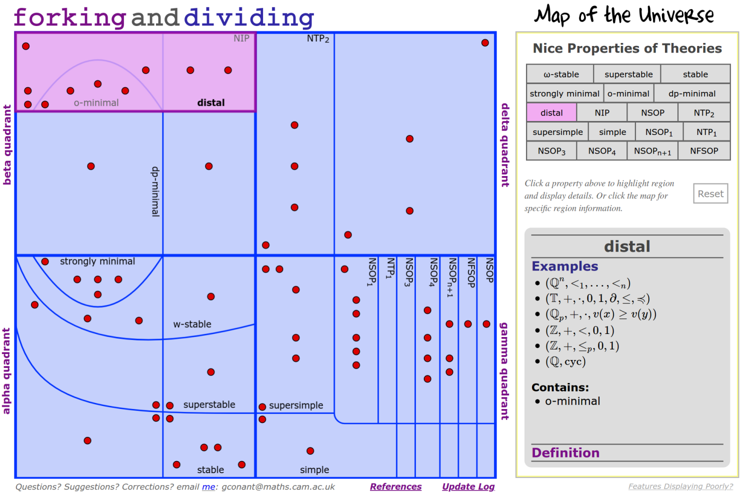

Figure: [4]

Distal Shatter Function, Exponent, Density

Definition

Given an abstract cell decomposition $\mathcal{T}$ for $\Phi(x; y)$:

- Let $\pi_{\mathcal{T}}(n) = \max_{|S|=n} |\mathcal{T}(S)|$
- Say that $\mathcal{T}$ *has exponent* d if $\pi_{\mathcal{T}}(n) = \mathcal{O}(n^d)$.
- Let the *distal density* of Φ be the infimum of all d such that there exists a distal cell decomposition for Φ of exponent d, or ∞ if no distal cell decomposition exists.

Fact

The distal density of Φ is at least the dual vc-density.

Corollary

All distal structures are NIP.

Distal Density for Certain Structures

$\mathcal{M}$	Dual VC-Density	Distal Density
o-minimal expansions of fields	$\lvert x\rvert$	$2\lvert x\rvert - 2$ (1 if $\lvert x\rvert = 1$)
weakly o-minimal structures	$\lvert x\rvert$	$2\lvert x\rvert - 1$
ordered vector spaces over ordered division rings	$\lvert x\rvert$	$\lvert x\rvert$
$\mathbb{Q}_p$ the valued field	$2\lvert x\rvert - 1$	$3\lvert x\rvert - 2$
$\mathbb{Q}_p$ in the linear reduct	$\lvert x\rvert$	$\lvert x\rvert$

(VC-Density Calculations: Aschenbrenner, Dolich, Haskell, Macpherson, Starchenko, except $\mathbb{Q}_p$ in the linear reduct by Bobkov)

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Distal Cutting Lemma

Definition

Given a real r and a family $A_1, \ldots, A_n$ of subsets of a set X, an *r-cutting* is a family $X_1, \ldots, X_t$ of sets such that

- $\bigcup_{i=1}^t X_i = X$.
- Each X_i is crossed by at most $\frac{n}{r}$ of the sets $A_1, \ldots, A_n$.

Theorem (Chernikov, Galvin, Starchenko)

Let $\varphi(x; y) \in \mathcal{L}$ be a formula admitting a distal cell decomposition $\mathcal{T}$ of exponent d.
Then for any finite $H \subseteq M^{|y|}$ of size n and any real r satisfying $1 < r < n$, the family $\{\varphi(M; a) : a \in H\}$ admits an r-cutting $X_1, \ldots, X_t$ with $t = \mathcal{O}(r^d)$.
Moreover, the X_is are intersections of at most two cells from $\mathcal{T}(H)$.

Zarankiewicz's Problem in Distal Structures

Theorem

Let $\mathcal{M}$ be a structure and $t \geq 2$. Assume that $E(x, y) \subseteq M^{|x|} \times M^{|y|}$ is a relation defined by a formula $\theta(x; y) \in \mathcal{L}$ that admits a distal cell decomposition of exponent t, and the graph $E(x, y)$ does not contain any $K_{s,u}$. Then for any finite $P \subseteq M^{|x|}, Q \subseteq M^{|y|}$, $|P| = m, |Q| = n$, we have:

$$|E(P, Q)| = \mathcal{O}\left(m^{\frac{(t-1)s}{ts-1}} n^{\frac{t(s-1)}{ts-1}} + m + n\right).$$

Zarankiewicz's Problem: Specific Cases

Corollary

Assume that $E(x,y) \subseteq \mathbb{R}^{|x|} \times \mathbb{R}^{|y|}$ is a relation given by a boolean combination of exponential-polynomial (in)equalities, and the graph $E(x,y)$ does not contain $K_{s,u}$. Then there is a constant $\alpha = \alpha(\theta, s, u)$ satisfying the following.
For any finite $P \subseteq \mathbb{R}^{|x|}, Q \subseteq \mathbb{R}^{|y|}$, $|P| = m, |Q| = n$, we have:

$$|E(P,Q)| \leq \alpha \left(m^{\frac{(2|x|-2)s}{(2|x|-1)s-1}} n^{\frac{(2|x|-1)(s-1)}{(2|x|-1)s-1}+\varepsilon} + m + n \right).$$

Zarankiewicz's Problem: Specific Cases

Corollary

Assume that $E(x, y) \subseteq \mathbb{Q}_p^{|x|} \times \mathbb{Q}_p^{|y|}$ is a relation given by a boolean combination of valuational (in)equalities between restricted analytic functions, and the graph $E(x, y)$ does not contain $K_{s,u}$. Then there is a constant $\alpha = \alpha(\theta, s, u)$ satisfying the following.
For any finite $P \subseteq \mathbb{Q}_p^{|x|}, Q \subseteq \mathbb{Q}_p^{|y|}$, $|P| = m, |Q| = n$, we have:

$$|E(P, Q)| \leq \alpha \left(m^{\frac{(3|x|-3)s}{(3|x|-2)s-1}} n^{\frac{(3|x|-2)(s-1)}{(3|x|-2)s-1}+\varepsilon} + m + n \right).$$

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Dimension Induction

Theorem (A.)

Let $\mathcal{M}$ be a structure in which all finite sets of formulas with $|x| = 1$ admit a distal cell decomposition $\mathcal{T}_1$ where every formula ψ of $\mathcal{T}_1$ refers to at most k parameters from S, and for some $d_0 \in \mathbb{N}$, all finite sets of formulas with $|x| = d_0$ have distal density at most r. Then all finite sets Φ of formulas with $|x| = d \geq d_0$ have distal density at most $k(d - d_0) + r$.

Dimension Induction Proof

The resulting cell decomposition generalizes the “vertical decomposition” of Chazelle, Edelsbrunner, Guibas, Sharir.

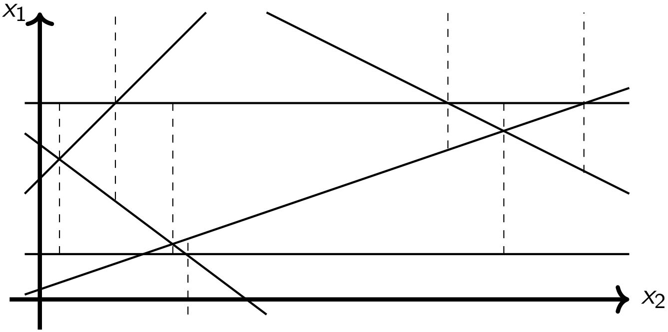

Figure: A “vertical decomposition” for lines in the plane[1]

Algorithmic Quantifier Elimination

- This construction is modeled after the “singly-exponential stratification” of semi-algebraic sets in [2].
- That cell decomposition can be used to algorithmically eliminate (existential) quantifiers, and Collins’s *cylindrical algebraic decomposition* can be used to eliminate all quantifiers for semi-algebraic sets.
- Distal cell decompositions may be useful for efficient QE in several distal structures.

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(Weakly) *o*-minimal structures

Definition

Let $\mathcal{M}$ be a structure whose language includes $<$, which is interpreted as a linear order.

- $\mathcal{M}$ is *weakly o-minimal* if every definable subset of M^1 is a finite union of $<$-convex pieces.
- $\mathcal{M}$ is *o-minimal* if those $<$-convex pieces are points and intervals.

DCDs in weakly *o*-minimal structures

Theorem (A.)

If $\mathcal{M}$ is a weakly o-minimal structure and $\Phi(x; y)$ is a finite set of formulas, then Φ has a distal cell decomposition of exponent $2|x| - 1$.

Proof.

We construct a distal cell decomposition $\mathcal{T}_1$ in the case $|x| = 1$ with 2 parameters, with $|\mathcal{T}_1(S)| = \mathcal{O}(|S|)$, and then apply Dimension Induction. $\square$

If $\mathcal{M}$ is an *o*-minimal expansion of a field, we can drop the exponent to $2|x| - 2$, using Chernikov, Galvin and Starchenko's bound in the plane.

Linear o-minimal case

Theorem

Let $\mathcal{M}$ be an ordered vector space over an ordered division ring A, in the language $\mathcal{L} = \{0, +, -, <, c\cdot_{c \in A}\}$. let $\Phi(x; y)$ be a finite set of formulas in the language of $\mathcal{M}$. Then Φ has a distal cell decomposition of exponent $|x|$.

The same bound also holds for any o-minimal locally modular expansion of an abelian group, as well as $\mathbb{Z}$ in Presburger's language.

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The Valued Field Structure

Let K be a P-minimal field (such as $\mathbb{Q}_p$) in the language $\mathcal{L}_{\mathrm{Mac}}$:

Definition

$\mathcal{L}_{\mathrm{Mac}} := \mathcal{L}_{\mathrm{ring}} \cup \{|, P_n : n \in \mathbb{N}\}$.

Describe the valuation by interpreting $x|y \iff v(x) \leq v(y)$.
Interpret P_n so that $P_n(x) \iff \exists y, y^n = x$.

Theorem (Macintyre)

K has quantifier-elimination.

Distal Cell Decomposition for K

Theorem

Let $\Phi(x; y)$ be a finite set of $\mathcal{L}_{\mathrm{Mac}}$-formulas. Then Φ admits a distal cell decomposition with exponent $3|x| - 2$.

Lemma

If $|x| = 1$, then Φ admits a distal cell decomposition with 3 parameters and exponent 1.

The Linear Reduct of $\mathbb{Q}_p$

Consider $\mathbb{Q}_p$ in the language $\mathcal{L}_{\mathrm{aff}}$:

Definition

$\mathcal{L}_{\mathrm{aff}} := \{0, 1, +, -, \{c\cdot\}_{c\in\mathbb{Q}_p}, |, \{Q_{m,n}\}_{m,n\in\mathbb{N}}\}.$

We interpret these symbols as the normal vector space structure, with $|$ describing the valuation as above, and $\mathbb{Q}_p \models Q_{m,n}(a) \iff a \in \bigcup_{k\in\mathbb{Z}} p^{km}(1 + p^n\mathbb{Z}_p)$.

Theorem (Leenknegt)

This structure on $\mathbb{Q}_p$ has quantifier elimination. It is also a reduct of the standard $\mathcal{L}_{\mathrm{Mac}}-$structure on $\mathbb{Q}_p$.

Distal Cell Decompositions in the Linear Reduct

Theorem (Bobkov)

If $\Phi(x; y)$ *is a finite set of* $\mathcal{L}_{\mathrm{aff}}$*-formulas, then* $\mathrm{vc}^*(\Phi) \leq |x|$.

Theorem (A.)

Φ *admits a distal cell decomposition of exponent* $|x|$.

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[2] Bernard Chazelle, Herbert Edelsbrunner, Leonidas J Guibas, and Micha Sharir. "A singly exponential stratification scheme for real semi-algebraic varieties and its applications". In: *Theoretical Computer Science* 83.1 (1991), pp. 77–105.

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