THE FRAÏSSÉ LIMIT OF MATRIX ALGEBRAS WITH THE RANK METRIC

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Abstract. We realize the $\mathbb{F}_q$-algebra $M(\mathbb{F}_q)$ studied by von Neumann and Halperin as the Fraïssé limit of the class of finite-dimensional matrix algebras over a finite field $\mathbb{F}_q$ equipped with the rank metric. We then provide a new Fraïssé-theoretic proof of uniqueness of such an object. Using the results of Carderi and Thom, we show that the automorphism group of $\mathrm{Aut}(\mathbb{F}_q)$ is extremely amenable. We deduce a Ramsey-theoretic property for the class of algebras $M(\mathbb{F}_q)$, and provide an explicit bound for the quantities involved.

1. Introduction

Fix q, a prime power. Let $\mathcal{K}(\mathbb{F}_q) = \{M_n(\mathbb{F}_q) : n \in \mathbb{N}\}$. If $m|n$, then let $\iota_{n,m}$ be the embedding of $M_m(\mathbb{F}_q)$ into $M_n(\mathbb{F}_q)$, given by $\iota_{n,m}(x) = x \otimes 1_{n/m}$. If $n_0, n_1, \ldots$ satisfies $n_k | n_{k+1}$ and $\lim_{k\to\infty} n_k = \infty$, then we call $n_0, n_1, \ldots$ a *factor sequence*, and

$$M_{n_0}(\mathbb{F}_q) \overset{\iota_{n_1,n_0}}{\hookrightarrow} M_{n_1}(\mathbb{F}_q) \overset{\iota_{n_2,n_1}}{\hookrightarrow} \ldots$$

is an inductive sequence of $\mathbb{F}_q$-algebras, so it has a direct limit, which we call $M_0(\mathbb{F}_q)$. Let ι_{n_k} be the corresponding inclusion of $M_{n_k}(\mathbb{F}_{\|})$ into $M_0(\mathbb{F}_q)$. On each $M_n(\mathbb{F}_q)$, we can define a metric, $d(x,y) = \frac{\mathrm{rank}(x-y)}{n}$. Under these metrics, each inclusion $\iota_{n,m}$ is an isometry, so a metric is induced on $M_0(\mathbb{F}_q)$. Let $M(\mathbb{F}_q)$ be the completion of $M_0(\mathbb{F}_q)$ under this metric, which is also an $\mathbb{F}_q$-algebra. In a manuscript eventually reworked and published by his student Halperin [9], von Neumann showed that $M(\mathbb{F}_q)$ is uniquely defined, that is, it does not depend on the choice of factor sequence.

In classical model theory, a Fraïssé class $\mathcal{K}$ is a collection of finitely-generated structures (or isomorphism classes thereof) in a given language, satisfying a few additional properties, which guarantee the existence and uniqueness of a Fraïssé limit associated with the class. The Fraïssé limit is a countably-generated structure F such that for any structure $A \in \mathcal{K}$, any isomorphism between substructures $A, B \in \mathcal{K}$ of F can be extended to an automorphism of F, a property known as $\mathcal{K}$-homogeneity [10]. This theory carries over to model theory of metric structures, where the limit need only be approximately $\mathcal{K}$-homogeneous [3]. In Section 2, we show that in the language of $\mathbb{F}_q$-algebras with a metric, the class $\mathcal{K}(\mathbb{F}_q)$ is a Fraïssé class, and in Section 4, we show that $M(\mathbb{F}_q)$ is its Fraïssé limit. We also use give a

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2000 *Mathematics Subject Classification.* Primary 16E50, 03C30; Secondary 03C98.

Key words and phrases. von Neumann regular ring, rank metric, Fraïssé class, Fraïssé limit.

A.A. was supported by Caltech's Summer Undergraduate Research Fellowships (SURF) program and by a Rose Hills Summer Undergraduate Research Fellowship. We would also like to thank Martino Lupini for mentorship and guidance throughout this project.

direct Fraïssé-theoretic proof of the uniqueness of $M(\mathbb{F}_q)$, originally established by von Neumann with a different argument.

Because of the added structure of the rank metric, there only exists an embedding from $M_m(\mathbb{F}_q)$ into $M_n(\mathbb{F}_q)$ if m divides n. However, in Sections 4 and 5, we wish to approximate embeddings from $M_m(\mathbb{F}_q)$ into $M(\mathbb{F}_q)$ by embeddings from $M_m(\mathbb{F}_q)$ into some finite stage of the direct limit defining $M(\mathbb{F}_q)$, say, $M_{n_k}(\mathbb{F}_q)$ for some k. To do this, unless some n_k is a multiple of m, we must instead approximate the embedding with approximate embeddings. In Section 3 we consider a natural notion of approximate embedding, and show that any embedding into $M(\mathbb{F}_q)$ can be approximated arbitrarily well by these approximate embeddings into finite stages of the limit. In order to establish this fact, we consider a presentation of $M_n(\mathbb{F}_q)$ in terms of generators and relations studied by Kassabov in [11], and prove that the defining relations are *stable* with respect to the rank metric. Stability problems for relations in metric groups and operator algebras have been estensively studied, also due to their connections with notions such as (linear) soficity and hyperlinearity in group theory, and (weak) semiprojectivity and $\mathcal{R}$-embeddability in operator algebras.

The Kechris-Pestov-Todorcevic correspondence establishes an equivalence between a *Ramsey property* of a Fraïssé class and *extreme amenability* of the automorphism group of its Fraïssé limit. The Ramsey property is a generalization of the Ramsey theorem, reducing to the standard Ramsey theorem for the Fraïssé class of finite linear orderings. Extreme amenability pertains to the topological dynamics of the group: a topological group G is extremely amenable when any continuous action of G on a compact Hausdorff space X leaves some point fixed [12]. This too carries over to metric Fraïssé structures, but again, the Ramsey property is only approximate [14]. In Section 5, we reduce the extreme amenability of $\mathrm{Aut}(M(\mathbb{F}_q))$ to the extreme amenability of the unit group of $M(\mathbb{F}_q)$, proven by Carderi and Thom [4].

It seems worth mentioning that the study of natural limiting objects of finite-dimensional matrix algebras has also connections with computer science and applied graph theory. In [13], the authors study Kronecker graphs, which are constructed by taking repeated tensor products of the adjacency matrices of graphs. By taking the tensor product sufficiently many times, one can construct a graph which is approximately self-similar, a process suitable for modelling fractal structures which appear in nature, or graphs such as social networks. However, in order to create a graph which would have genuine fractal structure, one would need to take the tensor product of an infinite sequence of matrices, which would no longer be a well-defined matrix. Such an object does however exist, as the limit of a Cauchy sequence of partial products, in the algebra $M(\mathbb{F}_q)$, so it may be possible to gain new insight into fractal graphs by studying this algebra further.

2. $\mathcal{K}(\mathbb{F}_q)$ is a Fraïssé class

Definition 2.1. Let $\mathcal{K}$ be a class of finitely-generated metric structures in a particular language. $\mathcal{K}$ is a metric Fraïssé class [6] if and only if the following properties are satisfied:

- Joint Embedding Property (JEP): For any $A, B \in \mathcal{K}$, there exists some $C \in \mathcal{K}$ such that A and B both embed into C.

- Near Amalgamation Property (NAP): For any $A, B_0, B_1 \in \mathcal{K}$, embeddings $\phi_i : A \hookrightarrow B_i$, and $\varepsilon > 0$, there exists some $C \in \mathcal{K}$ with embeddings $\psi_i : B_i \hookrightarrow C$ such that $d(\psi_0 \circ \phi_0, \psi_1 \circ \phi_1) < \varepsilon$.
- Weak Polish Property (WPP): For any class satisfying JEP and NAP, we can define, for each $n \in \mathbb{N}$, a class $\mathcal{K}_n$ of structures in $\mathcal{K}$, with specified generating tuples of size at most n. We then define a pseudometric on $\mathcal{K}_n$ (relying on JEP and NAP) by
$$d_n(\bar{a}, \bar{b}) = \inf d(\phi(\bar{a}), \psi(\bar{b}))$$
where $\phi : \langle \bar{a} \rangle \hookrightarrow C, \psi : \langle \bar{b} \rangle \hookrightarrow C$ are embeddings into the same structure $C \in \mathcal{K}$. The WPP is true when each of these pseudometrics d_n is separable.

We will now verify that $\mathcal{K}(\mathbb{F}_q)$ satisfies each of these properties, and is thus a Fraïssé class, implying the existence of a unique Fraïssé limit.

Joint Embedding Property. If $m|n$, then let $\iota_{n,m}$ be the embedding of $M_m(\mathbb{F}_q)$ into $M_n(\mathbb{F}_q)$, given by $\iota_{n,m}(x) = x \otimes 1_{n/m}$ [5].

Let A and B be structures in $\mathcal{K}(\mathbb{F}_q)$, that is, $A = M_a(\mathbb{F}_q)$ and $B = M_b(\mathbb{F}_q)$. Then if $C = M_{ab}(\mathbb{F}_q)$, there exists an embedding $\iota_{ab,a} : A \hookrightarrow C$, and an embedding $\iota_{ab,b} : B \hookrightarrow C$, so A and B can be jointly embedded into C.

Amalgamation Property. In this case, $\mathcal{K}(\mathbb{F}_q)$ satisfies not only the Near Amalgamation Property, but the same amalgamation property as discrete structures, allowing us to dispense with the ε: for any $A, B_0, B_1 \in \mathcal{K}$, embeddings $\phi_i : A \hookrightarrow B_i$, there exists some $C \in \mathcal{K}$ with embeddings $\psi_i : B_i \hookrightarrow C$ such that $\psi_0 \circ \phi_0 = \psi_1 \circ \phi_1$.

Let A, B_0, B_1 be structures in $\mathcal{K}(\mathbb{F}_q)$, with embeddings $\phi_i : A \to B_i$. Let $A = M_a(\mathbb{F}_q)$ and $B_i = M_{b_i}(\mathbb{F}_q)$. As A, B_i are matrix algebras over $\mathbb{F}_q$, and thus finite-dimensional central simple algebras over $\mathbb{F}_q$, the Skolem-Noether Theorem [7] shows that each embedding $\phi_i : A \hookrightarrow B_i$ must be a composition of $\iota_{b_i,a}$ with an inner automorphism of B_i, given by conjugating by some unit $y_i \in B_i^*$. Thus we may assume without loss of generality that each $\phi_i = \iota_{b_i,a}$.

Thus if $C = M_c(\mathbb{F}_q)$, where b_0, b_1 both divide c, we can use the automorphisms ι_{c,b_i} to make the following diagram commute:

$$\begin{tikzcd}
 & M_c(\mathbb{F}_q) & \\
M_{b_0}(\mathbb{F}_q) \arrow[ur, "\iota_{c,b_0}"] & & M_{b_1}(\mathbb{F}_q) \arrow[ul, "\iota_{c,b_1}"'] \\
 & M_a(\mathbb{F}_q) \arrow[ul, "\iota_{b_0,a}"] \arrow[ur, "\iota_{b_1,a}"'] &
\end{tikzcd}$$

This commutes because for any i, j, k,
$$\iota_{ijk,ij} \circ \iota_{ij,i}(x) = \iota_{ij,i}(x) \otimes 1_k = x \otimes 1_j \otimes 1_k = x \otimes 1_{jk} = \iota_{ijk,i}(x)$$
and thus $\iota_{b_0 b_1, b_i} \circ \iota_{b_i,a} = \iota_{b_0 b_1, a}$ for each i.

Weak Polish Property. There are only countably many structures in $\mathcal{K}(\mathbb{F}_q)$, and each one is finite. Thus each $\mathcal{K}(\mathbb{F}_q)_n$ is countable, and thus trivially separable.

3. Stability Lemma

3.1. δ-embeddings.

Definition 3.1. Define a (not necessarily unital) homomorphism $\phi : M_m(\mathbb{F}_q) \to M_n(\mathbb{F}_q)$ to be a δ*-embedding* when there exists some unit $y \in M_n(\mathbb{F}_q)$ and some number k such that for each $x \in M_m(\mathbb{F}_q)$, $\phi(x) = y(x^{\oplus k} \oplus 0^{n-mk})y^{-1}$, and $\frac{mk}{n} \geq 1 - \delta$.

δ-embeddings will be used as a proxy for actual embeddings, because we cannot always guarantee that there will be an embedding $M_m(\mathbb{F}_q) \hookrightarrow M_n(\mathbb{F}_q)$, unless we know that m divides n. We can reconstruct an actual embedding by taking a limit of δ-embeddings with δ decreasing to 0.

3.2. **Proving the Lemma.**

Lemma 3.1. *Let $M(\mathbb{F}_q)$ be the completion of the direct limit of the sequence $M_{n_0}(\mathbb{F}_q) \hookrightarrow M_{n_1}(\mathbb{F}_q) \hookrightarrow \dots$, and let $\phi : M_n(\mathbb{F}_q) \hookrightarrow M(\mathbb{F}_q)$ be an embedding. Then for each $\varepsilon, \delta > 0$, there exists some N such that if $K \geq N$, if $n_K = mn + r$ with $0 \leq r < n$, then there is a $\frac{r}{n_K}$-embedding $\psi : M_n(\mathbb{F}_q) \to M_{n_K}(\mathbb{F}_q)$ such that $d(\iota_{n_K} \circ \psi, \phi) < \varepsilon$. In particular, if n divides n_K, then ψ is an embedding.*

Proof. Fix $\varepsilon > 0$. As established in [11], for a prime p, $M_n(\mathbb{F}_p)$ is the ring presented by the following generators and relations

$$M_n(\mathbb{F}_p) = \langle a, b | a^n = b^n = 0, ba + (p+1)a^{n-1}b^{n-1} = 1 \rangle$$

where a and b are the off-diagonal matrices

$$\begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & & & & \vdots \\ 0 & 1 & 0 & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & 1 & 0 & & \vdots \\ \vdots & & & & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & & & & \vdots \\ \vdots & & 0 & 1 & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & 0 & 1 & 0 \\ \vdots & & & & & & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

Choose p to be the prime such that $p^k = q$. The set of matrices in $M_n(\mathbb{F}_q)$ with coordinates from $\mathbb{F}_p$ is an isomorphic copy of $M_n(\mathbb{F}_p)$, so we can find $a, b \in M_n(\mathbb{F}_q)$ that generate the embedded copy of $M_n(\mathbb{F}_p)$ as a ring. As the embedded copy of $M_n(\mathbb{F}_p)$ contains a basis of $M_n(\mathbb{F}_q)$ as a $\mathbb{F}_q$-vector space, a, b generate $M_n(\mathbb{F}_q)$ as a $\mathbb{F}_q$-algebra.

Now we look at $\phi(M_n(\mathbb{F}_q))$, and in particular, its generators $\phi(a), \phi(b)$. We can write $\phi(a)$ as the limit of a Cauchy sequence $a_1, a_2, \dots$ and $\phi(b)$ as the limit of a Cauchy sequence $b_1, b_2, \dots$, with each a_i, b_i in the direct limit of $M_{n_0}(\mathbb{F}_q) \hookrightarrow M_{n_1}(\mathbb{F}_q) \hookrightarrow \dots$. Let $\delta > 0$. Then as the operations $+, \cdot, d$ are continuous, we can find some a_i, b_j such that

$$d(a_i^n, 0), d(b_j^n, 0), d(a_i, \phi(a)), d(b_j, \phi(b)), d(b_j a_i + a_i^{n-1} b_j^{n-1}, 1) < \delta$$

and

$$\left| d(a_i, 0) - \frac{n-1}{n} \right|, \left| d(b_j, 0) - \frac{n-1}{n} \right| < \delta.$$

Let N be such that a_i, b_j are both in the image $\iota_{n_N}(M_{n_N}(\mathbb{F}_q))$. Let $K \geq N$ and note that a_i, b_j are both in the image $\iota_{n_K}(M_{n_K}(\mathbb{F}_q))$. Let $x = \iota_{n_K}^{-1}(a_i)$ and $y = \iota_{n_K}^{-1}(b_j)$ be matrices acting on the vector space $\mathbb{F}_q^{n_K}$. Then let $W_0 = \ker y \cap \ker x^n \cap \ker(yx + x^{n-1}y^{n-1} - 1)$. Clearly $\dim\ker y = n_K - \operatorname{rank}(y)$, so $\left|\frac{\dim\ker y}{n_K} - \frac{1}{n}\right| < \delta$, and as each of the other operators has rank at most $n_K\delta$, each of their kernels has dimension at least $n_K(1-\delta)$, and W_0, the intersection of these three kernels, has dimension at least $n_K(\frac{1}{n} - 3\delta)$.

As $W_0 \subset \ker(yx + x^{n-1}y^{n-1} - 1) \cap \ker y$, we know that for any $v \in W_0$,

$$(yx + x^{n-1}y^{n-1})v = yxv = v$$

so on the restricted domain of W_0, $yx = 1$. Thus x maps W_0 isomorphically onto xW_0 with inverse y. For any $1 \leq k \leq n-1$, if $W_{k-1} \subset \ker(yx-1) \cap \ker y^k$ has been defined, let $W_k = xW_{k-1} \cap \ker(yx + x^{n-1}y^{n-1} - 1)$. Then as yx is the identity on W_{k-1}, x is an isomorphism from W_{k-1} to xW_{k-1} with inverse y. Thus y is an isomorphism from W_k to $yW_k \subset W_{k-1} \subset \ker y^k$, so $W_k \subset \ker y^{k+1}$. If $k < n-1$, we also have $0 = yx + x^{n-1}y^{n-1} - 1 = yx - 1$ on W_k, so $W_k \in \ker(yx-1) \cap \ker y^{k+1}$, satisfying the inductive hypothesis for the next step. Thus we can apply this recursive definition all the way through W_{n-1}, provided the base case works. As $W_0 \subset \ker(yx-1) \cap \ker y^1$, the base case checks out, and this recursive definition is well-defined.

Also, as $\dim xW_{k-1} = \dim W_{k-1}$ and $yx + x^{n-1}y^{n-1} - 1$ has rank at most $n_K\delta$,

$$\dim W_k \geq \dim xW_{k-1} - n_K\delta = \dim W_{k-1} - n_K\delta.$$

As $\dim W_0 \geq n_K(\frac{1}{n} - 4\delta)$, we have that $\dim W_k \geq n_K(\frac{1}{n} - (4+k)\delta) \geq n_K(\frac{1}{n} - (4+n)\delta)$. Now we define $V = W_{n-1} + yW_{n-1} + \cdots + y^{n-1}W_{n-1}$, and we wish to prove that $V \subset \ker x^n \cap \ker y^n \cap \ker(yx + x^{n-1}y^{n-1} - 1)$. For any $0 \leq k \leq n-1$, as $y^{n-1}W_{n-1} \subset W_0 \subset \ker y$, $y^k W_{n-1} \subset \ker y^{n-k}$, so $V \subset \ker y^n$. For any $0 \leq k \leq n-1$, $y^k W_{n-1} \subset x^{n-k-1}W_0$. As $W_0 \subset \ker x^n$, we know that $y^k W_{n-1} \subset x^{n-k-1}W_0 \subset \ker x^{k+1} \subset \ker x^n$, and $V \subset \ker x^n$. Thus also $W_{n-1} \subset \ker x \subset \ker yx$, and as xy is the identity on W_k for any $k > 0$, $x^k y = x^{k-1}$ on W_k, which contains $y^{k-1}W_{n-1}$, so as xy is the identity on W_1, by induction, $x^k y^k$ is the identity on $y^{n-k-1}W_{n-1}$, and $x^{n-1}y^{n-1}$ is the identity on W_{n-1}. Thus $yx + x^{n-1}y^{n-1} = 1$ on W_{n-1}, and $W_{n-1} \subset \ker(yx + x^{n-1}y^{n-1} - 1)$. We now need to show that $y^k W_{n-1} \subset \ker(yx + x^{n-1}y^{n-1} - 1)$ for $k \geq 1$. In that case, $y^k W_{n-1} \subset \ker y^{n-k} \subset y^{n-1}$, and by assumption, $y^k W_{n-1} \subset W_{n-k-1} \subset \ker(yx-1)$, so $yx + x^{n-1}y^{n-1} - 1 = yx - 1 = 0$ on $y^k W_{n-1}$ as desired.

Thus $V \subset \ker x^n \cap \ker y^n \cap \ker(yx + x^{n-1}y^{n-1} - 1)$, and the relations $x^n = y^n = 0 = yx + x^{n-1}y^{n-1} - 1$ are satisfied when restricted to the domain V. Thus on the domain V, for some unit $B \in M_{n_K}(\mathbb{F}_q)$ representing a change-of-basis, $x = B(a^{\oplus \frac{\dim V}{n}} \oplus 0^{\oplus n_K - \dim V})B^{-1}$ and $y = B(b^{\oplus \frac{\dim V}{n}} \oplus 0^{\oplus n_K - \dim V})B^{-1}$.

Let $n_K = nm + r$, with $0 \leq r < n$. Then as n divides $\dim V$, $\frac{\dim V}{n} \leq m$. If we let $x'^{\oplus m} \oplus 0^{\oplus r})B^{-1}$ and $y = B(b^{\oplus m} \oplus 0^{\oplus r})B^{-1}$, we see that $x'^{\oplus \frac{\dim V}{n}} a^{\oplus m - \frac{\dim V}{n}} \oplus 0^{\oplus r})B^{-1}$ and $y'^{\oplus \frac{\dim V}{n}} b^{\oplus m - \frac{\dim V}{n}} \oplus 0^{\oplus r})B^{-1}$, so both of these clearly have rank

$$m - \frac{\dim V}{n} \leq \frac{n_K - \dim V}{n}$$

Thus if we define the homomorphism ψ on the generators by $\psi(a) = x'$ and $\psi(b) = y'$, we find that ψ is a $\frac{r}{n_K}$-embedding, and it suffices to show that $d(\iota_{n_K} \circ \psi, \phi) < \varepsilon$.

To do this, it suffices to show that $d(\iota_{n_K} \circ \psi(a), \phi(a)), d(\iota_{n_K} \circ \psi(b), \phi(b)) < \gamma$ for some $\gamma < 0$ depending on ε. As

$$d(\iota_{n_K} \circ \psi(a), \phi(a)) = d(\iota_{n_K}(x'), \phi(a)) \leq d(x, x') + d(\iota_{n_K}(x), \phi(a)) \leq d(x, x') + \delta,$$

and similarly $d(\iota_{n_K} \circ \psi(b), \phi(b)) \leq d(y, y') + \delta$, we recall that $d(x, x'), d(y, y') \leq \frac{n_K - \dim V}{n}$, so we only need to show that $\frac{n_K - \dim V}{n} + \delta < \gamma$ for sufficiently small δ.

For $0 \leq r < s \leq n$, we will show that $y^r W_{n-1} \cap y^s W_{n-1} = 0$. As $y^r W_{n-1} \subset \ker x^{r+1}$ and $y^s W_{n-1} \subset \ker y^{n-s}$, if $v \in y^r W_{n-1} \cap y^s W_{n-1}$, then $v \in \ker x^{r+1} \cap y^{n-s}$. As $\langle x, y \rangle \cong M_n(\mathbb{F}_q)$ when restricted to the domain V, $x^s y^s + y^{n-s} x^{n-s} = 1$ on V, so $v = (y^s x^s + x^{n-s} y^{n-s})v = 0$. Thus $y^r W_{n-1} \cap y^s W_{n-1} = 0$, and

$$\dim V = \sum_{s=0}^{n-1} \dim y^s W_{n-1} = n \dim W_{n-1} \geq n_K(1 - (4 + n)n\delta)$$

Placing this in our earlier inequality, we find that

$$d(x, x'), d(y, y') < \frac{n_K - \dim V}{n_K} \leq (4 + n)n\delta$$

so by taking δ low enough, we find

$$d(x, x') + \delta, d(y, y'^2 + 4n + 1)\delta < \gamma$$

as desired. □

4. Explicit Fraïssé Theory

As $\mathcal{K}(\mathbb{F}_q)$ is a Fraïssé class, it must have a unique Fraïssé limit. A Fraïssé limit of $\mathcal{K}(\mathbb{F}_q)$ is a $\mathcal{K}(\mathbb{F}_q)$-*structure* which is $\mathcal{K}(\mathbb{F}_q)$-*universal* and *approximately homogeneous* [6]. A $\mathcal{K}(\mathbb{F}_q)$-structure is a structure which can be realized as the direct limit (in the category of metric structures with the appropriate signature) of a sequence of elements of $\mathcal{K}(\mathbb{F}_q)$, which is, in this case, the completion of the algebraic direct limit of an inductive sequence of elements of $\mathcal{K}(\mathbb{F}_q)$, or $M(\mathbb{F}_q)$ for some factor sequence. Given von Neumann's result, it is clear that there is only one $\mathcal{K}(\mathbb{F}_q)$-structure up to isomorphism, so this must be the Fraïssé limit. If we do not assume this result, we can still use the uniqueness of the Fraïssé limit to directly prove the uniqueness of $M(\mathbb{F}_q)$.

First we will show that if the factor sequence $n_0|n_1|\ldots$ is given by $n_i = i!$, then the completion $M(\mathbb{F}_q)$ of the corresponding direct limit is a Fraïssé limit. Then we will show, with a back-and-forth argument mirroring the classic proof of the uniqueness of the Fraïssé limit, that all $\mathcal{K}(\mathbb{F}_q)$-structures are isomorphic to $M(\mathbb{F}_q)$.[10]

The factor sequence $0!, 1!, 2!, \ldots$ is chosen to make $\mathcal{K}$-universality simple to prove. For any i, we know an embedding of $M_i(\mathbb{F}_q)$ into $M_{i!}(\mathbb{F}_q)$, as i divides $i!$. Thus $M_i(\mathbb{F}_q)$ embeds into $M_{n_i}(\mathbb{F}_q)$, and thus into $M(\mathbb{F}_q)$.

4.1. **Approximate Homogeneity.**

Fix a factor sequence $n_0|n_1|\ldots$. We will show that the completion of its direct limit, $M(\mathbb{F}_q)$, is the Fraïssé limit of all $M_n(\mathbb{F}_q)$s. Let $\phi, \psi : M_n(\mathbb{F}_q) \hookrightarrow M(\mathbb{F}_q)$ be embeddings. Fix $\varepsilon > 0$. Then we apply Lemma 3.1 to both ϕ, ψ, letting N_ϕ be the value of N that suffices for ϕ, and N_ψ the value of N that suffices for ψ. Then let $K = \max N_\phi, N_\psi$. By the choice of N_ϕ and N_ψ, we see that there is a ring homomorphism $\phi' : M_n(\mathbb{F}_q) \to M_{n_K}(\mathbb{F}_q)$ with $d(\iota_{n_K} \circ \phi', \phi) < \frac{\varepsilon}{2}$, and similarly a homomorphism ψ' close to ψ, together with units $B_\phi, B_\psi \in M_{n_K}(\mathbb{F}_q)$ such that where $n_K = nm + r$ and $0 \leq r < n$, for all $A \in M_n(\mathbb{F}_q)$,

$\phi'(A) = B_\phi(A^{\oplus m} \oplus 0^{\oplus r})B_\phi^{-1}$, and similarly for ψ'. Thus $B_\psi B_\phi^{-1}\phi'(A)B_\phi B_\psi^{-1} = B_\psi(A^{\oplus m} \oplus 0^{\oplus r})B_\psi^{-1} = \psi'(A)$. Thus if β is the inner automorphism given by conjugation by $B_\psi B_\phi^{-1}$, we have $\psi' = \beta \circ \phi'$, and for $A \in M_n(\mathbb{F}_q)$,

$$\begin{aligned} d(\beta \circ \phi(A), \psi(A)) &\leq d(\beta \circ \phi(A), \beta \circ \phi'(A)) + d(\psi(A), \psi'(A)) \\ &= d(\phi(A), \phi'(A)) + d(\psi(A), \psi'(A)) \\ &< \varepsilon \end{aligned}$$

so $d(\beta \circ \phi, \psi) < \varepsilon$.

4.2. **Uniqueness.** The Fraïssé limit is also unique, at least among $\mathcal{K}(\mathbb{F}_q)$-structures, which are direct limits of elements of $\mathcal{K}(\mathbb{F}_q)$. [6] In fact, as von Neumann and Halperin showed, there is only one $\mathcal{K}(\mathbb{F}_q)$-structure, $M(\mathbb{F}_q)$ (as metric structures must be complete, the direct limit of an inductive sequence of metric $\mathbb{F}_q$-algebras is the completion of the algebraic direct limit). We shall provide an alternate proof of this fact, following the classic proof of the uniqueness of the Fraïssé limit.

First we show the approximate extension property.

Lemma 4.1 (Approximate Extension Property)**.** *Let $m_0, m_1, \ldots$ be a factor sequence. Fix $\delta, \delta' > 0$, and let $\phi : M_{m_k}(\mathbb{F}_q) \to M_n(\mathbb{F}_q)$ be a δ-embedding. There exists some $k' \geq k$ and a δ'-embedding $\psi : M_n(\mathbb{F}_q) \to M_{m_{k'}}(\mathbb{F}_q)$ such that the following diagram commutes up to $\delta + \delta'$:*

$$\begin{tikzcd} M_{m_{k'}}(\mathbb{F}_q) & M_n(\mathbb{F}_q) \arrow[l, "\psi"'] \\ M_{m_k}(\mathbb{F}_q) \arrow[u, hook, "i_{m_{k'},m_k}"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proof. As ϕ is a δ-embedding, we can write it as $\phi : a \mapsto y(a^{\oplus r} \oplus 0^{n-m_k r})y^{-1}$ where $m_k r \geq (1-\delta)n$. Let $\psi : b \mapsto z((y^{-1}by)^{\oplus s} \oplus 0^{m_{k'}-sn})z^{-1}$, which makes ψ a δ'-embedding as long as $\frac{m_{k'}-sn}{m_{k'}} \leq \delta'$ which is satisfied when $\delta' m_{k'} > n$, and $s = \lfloor \frac{m_{k'}}{n} \rfloor$, so $m_{k'} - sn < n < \delta' m_{k'}$.

Then the only remaining requirement is that for all $a \in M_{m_k}(\mathbb{F}_q)$, $d(\psi \circ \phi(a), i_{m_{k'},m_k}(a)) > \delta + \delta'$.

$$\begin{aligned} \psi \circ \phi(a) &= z((y^{-1}y(a^{\oplus r} \oplus 0^{n-m_k r})y^{-1}y)^{\oplus s} \oplus 0^{m_{k'}-sn})z^{-1} \\ &\leq z((a^{\oplus r} \oplus 0^{n-m_k r})^{\oplus s} \oplus 0^{m_{k'}-sn})z^{-1} \end{aligned}$$

so with the correct choice of z,

$$\psi \circ \phi(a) = a^{\oplus rs} \oplus 0^{m_{k'}-rsm_k}$$

so

$$d(\psi \circ \phi(a), i_{m_{k'},m_k}(a)) = 1 - \frac{rsm_k}{m_{k'}}$$

and

$$\frac{rsm_k}{m_{k'}} \geq (1-\delta)\frac{sn}{m_{k'}} \geq (1-\delta)(1-\delta') > 1 - (\delta + \delta')$$

so we have

$$d(\psi \circ \phi(a), i_{m_{k'},m_k}(a)) = 1 - \frac{rsm_k}{m_{k'}} < \delta + \delta'$$

as desired. $\square$

Theorem 4.1. *If X and Y are $\mathcal{K}(\mathbb{F}_q)$-structures, then $X \cong Y$.*

Proof. Let X be the completion of a direct limit corresponding to the from the factor sequence $m_0, m_1, \dots$, and let Y be the completion of the direct limit corresponding to the factor sequence $n_0, n_1, \dots$. Let $\phi_0 : M_{m_0}(\mathbb{F}_q) \to M_{n_0}(\mathbb{F}_q)$ be a 1-embedding, so $i_{n_1,n_0} \circ \phi_0$ is a 1-embedding as well. Thus by Lemma 4.1, there is some $M_{m_{j_1}}(\mathbb{F}_q)$ and a 2^{-1}-embedding $\psi_0 : M_{n_0}(\mathbb{F}_q) \to M_{m_{j_1}}$ such that $d(\psi_0 \circ i_{n_1,n_0} \circ \phi_0, i_{m_{j_1},m_0}) < 1+2^{-1}$. We define $j_0 = k_0 = 0$, and given j_i or k_i, define $X_i = M_{m_{j_i}}$ and $Y_i = M_{n_{k_i}}$. Now we continue this process recursively. Let $\phi_i : X_{2i} \to Y_{2i}$ is a 2^{-2i}-embedding. We define $k_{2i+1} = k_{2i} + 1$, so that the sequence $Y_0, Y_1, \dots$ does not terminate. By Lemma 4.1, there is a X_{2i+1} and a $2^{-(2i+1)}$-embedding $\psi_i : Y_{2i+1} \to X_{2i+1}$ such that $d(\psi_i \circ \iota_{n_{k_{2i+1}},n_{k_{2i}}} \circ \phi_i, \iota_{m_{j_{2i+1}},m_{j_{2i}}}) < 2^{-2i} + 2^{-(2i+1)}$. This just generalizes the case of $i = 0$.

Similarly, let $\psi_i : Y_{2i+1} \to X_{2i+1}$ be a $2^{-(2i+1)}$-embedding. Then we define $j_{2i+2} = j_{2i+1} + 1$, so that the sequence $X_0, X_1, \dots$ does not terminate either. By Lemma 4.1, there is a Y_{2i+2} and a $2^{-(2i+2)}$-embedding $\phi_{i+1} : X_{2i+2} \to Y_{2i+2}$ such that $d(\phi_{i+1} \circ \psi_i, \mathrm{id}) < 2^{-(2i+1)} + 2^{-(2i+2)}$.

We will show that $d(\phi_n(x), \phi_{n+1}(x)) < 2^{-n+2}$.

$$
\begin{aligned}
&d(\phi_n(x), \phi_{n+1}(x)) \\
&\qquad\qquad \le d(\phi_n(x), \phi_{n+1} \circ \psi_n \circ \phi_n(x)) + d(\phi_{n+1} \circ \psi_n \circ \phi_n(x), \phi_{n+1}(x)) \\
&\qquad\qquad \le d(\phi_n(x), \phi_{n+1} \circ \psi_n \circ \phi_n(x)) + d(\psi_n \circ \phi_n(x), x) \\
&\qquad\qquad < (2^{-2n} + 2^{-2n-1}) + (2^{-2n-2} + 2^{-2n-3}) < 2^{-2n+1}
\end{aligned}
$$

and thus if $n < m$, $d(\phi_m(x), \phi_n(x)) < \sum_{k=n}^{m-1} 2^{-2k+1} < 2^{-2n+2}$, and $\phi_n(x), \phi_{n+1}(x), \dots$ is a Cauchy Sequence, and by the same proof so is $\psi_n(y), \dots$. Now we define $\phi = \lim_i \phi_i$ and $\psi = \lim_i \psi_i$ pointwise on $\bigcup_i X_i$ and $\bigcup_i Y_i$ respectively. Then for any i and $x \in X_{2i}$, $d(\psi_i \circ \phi_i(x), x) < 2^{-2i} + 2^{-2i-1}$. Thus if $x \in X_{2i}$, $\lim_{j\to\infty} d(\psi_j \circ \phi_j(x), x) = 0$.

For any δ-embedding $\theta : A \to B$, $d(\theta(1_A), 1_B) < \delta$, so $\phi(1) = \lim_i \phi_i(1) = 1$, and by the same reasoning, ψ is also unital.

Now we show that ψ, ϕ are 1-Lipschitz. Fix $\varepsilon > 0$, and let $x_1, x_2 \in \bigcup_i X_i$ be such that $d(x_1, x_2) < \varepsilon$. Let j be such that $x_1, x_2 \in X_{2n}$. Then if $r(x)$ is the normalized rank of x, we have for any $x \in X_{2n}$, $r(\phi_n(x)) \le r(x)$ because ϕ_j is a δ-embedding for some δ, and thus $d(\phi_n(x_1), \phi_n(x_2)) \le d(x_1, x_2) < \varepsilon$, so ϕ is 1-Lipschitz on $\bigcup_i X_i$, which is dense in X, so it is possible to extend ϕ to the entirety of X as a 1-Lipschitz and thus continuous map. Similarly, ψ is 1-Lipschitz, and can be extended to all of Y.

Fix $x_1, x_2 \in \bigcup_i X_i$, and any $\delta > 0$, there is some N such that if $n > N$, ϕ_n is a δ-embedding. Thus also $d(\phi_n(x_1), \phi_n(x_2)) \ge (1 - \delta) d(x_1, x_2)$, so $d(\phi(x_1), \phi(x_2)) = d(x_1, x_2)$, and ϕ (and similarly ψ) is an isometry.

We now wish to show that $\psi \circ \phi$ is the identity on X, and the same proof will show that $\phi \circ \psi$ is the identity on Y. We note that for any $n < m$, as proven earlier,

$d(\phi_m(x), \phi_n(x)) < 2^{-n+2}$, and $d(\psi_m(y), \psi_n(y)) < 2^{-2n+1}$. Thus for any n,

$$\begin{aligned}
&d(\psi\circ\phi(x), x) \\
&\qquad \le d(\psi\circ\phi(x), \psi\circ\phi_n(x)) + d(\psi\circ\phi_n(x), \psi_n\circ\phi_n(x)) + d(\psi_n\circ\phi_n(x), x) \\
&\qquad < d(\phi(x), \phi_n(x)) + d(\psi\circ\phi_n(x), \psi_n\circ\phi_n(x)) + (2^{-2n} + 2^{-2n-1}) \\
&\qquad < 2^{-2n+2} + 2^{-2n+1} + 2^{-2n} + 2^{-2n-1} \\
&\qquad < 2^{-2n+3}
\end{aligned}$$

and therefore $d(\psi\circ\phi(x), x) = 0$, as desired.

As ψ and ϕ are inverses, and each is a unital isometric homomorphism, they are isomorphisms, and $X \cong Y$. $\square$

5. Extreme Amenability

5.1. **The set of inner automorphisms is dense in $\mathrm{Aut}(M(\mathbb{F}_q))$.** In order to show that the inner automorphisms are dense in $\mathrm{Aut}(M(\mathbb{F}_q))$, it suffices to choose an automorphism $\phi \in \mathrm{Aut}(M(\mathbb{F}_q))$, and a basis open neighborhood around it, and find an inner automorphism in that neighborhood. Fix a factor sequence $n_0, n_1, \dots,$ and let $M_0(\mathbb{F}_q)$ be the direct limit associated to it, dense in $M(\mathbb{F}_q)$. Now let U be a basis neighborhood around ϕ, which will be of the form $\bigcap_{x\in X}\{f \in \mathrm{Aut}(M(\mathbb{F}_q)) : d(f(x), \phi(x)) < \varepsilon\}$ for some finite set $X = \{x_1, \dots, x_k\}$.

For each $x_i \in X$, let $y_i \in M_0(\mathbb{F}_q)$ be such that $d(x_i, y_i) < \frac{\varepsilon}{3}$. Then we shall find an inner automorphism ψ such that $d(\psi(y_i), \phi(y_i)) < \frac{\varepsilon}{3}$ for each y_i. Given such an automorphism, we find

$$\begin{aligned}
d(\psi(x_i), \phi(x_i)) &< d(\psi(x_i), \psi(y_i)) + d(\psi(y_i), \phi(y_i)) + d(\phi(y_i), \phi(x_i)) \\
&< 2d(x_i, y_i) + \frac{\varepsilon}{3} \\
&< \varepsilon
\end{aligned}$$

so $\psi \in U$. Thus it suffices to find an inner automorphism ψ such that $d(\psi(y), \phi(y)) < \varepsilon$ for each $y \in Y$, for each finite set $Y \subset M_0(\mathbb{F}_q)$ and each $\varepsilon > 0$.

Let us fix some such $Y \subset M_0(\mathbb{F}_q)$ and $\varepsilon > 0$. As Y is finite, there must be some n_m such that Y is contained in the image $\iota_{n_m}(M_{n_m}(\mathbb{F}_q))$.

Define $\phi' : M_{n_m}(\mathbb{F}_q) \hookrightarrow M(\mathbb{F}_q)$ by $\phi' = \phi\circ\iota_{n_m}$. Clearly ϕ' is an embedding. By Lemma 3.1, there exists some n_K, and an embedding $\psi : M_{n_m}(\mathbb{F}_q) \hookrightarrow M_{n_K}(\mathbb{F}_q)$ such that $d(\iota_{n_K}\circ\psi(x), \phi'(x)) < \varepsilon$ for each $x \in M_{n_m}(\mathbb{F}_q)$, so for each $y \in Y$, as $Y \subset M_{n_m}(\mathbb{F}_q)$, $d(\iota_{n_K}\circ\psi(y), \phi'(y)) < \varepsilon$.

5.2. **Quotient.** Let $A(p)$ be as in the paper by Carderi and Thom.[4] They assert that $A(p)$ is the group of units of $M(\mathbb{F}_q)$, which obviously has a continuous surjective homomorphism onto the inner automorphism group $B(p)$ of units of $M(\mathbb{F}_q)$. As $B(p)$, the image of $A(p)$ under a continuous homomorphism, is dense in $\mathrm{Aut}(M(\mathbb{F}_q))$, we have, with Proposition 6.2 of [8], that $\mathrm{Aut}(M(\mathbb{F}_q))$ is itself extremely amenable, and by the KPT correspondence, the class of matrix algebras has the Ramsey Property.

6. Ramsey Property

Theorem 6.1. *The Fraïssé class $\mathcal{K}(\mathbb{F}_q)$ has the approximate Ramsey Property. That is, if $A, B \in \mathcal{K}(\mathbb{F}_q)$, and $\varepsilon > 0$, there exists some $C \in \mathcal{K}(\mathbb{F}_q)$ such that for any* *continuous coloring γ of $\binom{C}{A}$, that is, a 1-Lipschitz map $\gamma : \binom{C}{A} \to [0,1]$, there is some $B' \in \binom{C}{B}$ such that the oscillation of γ over the subset $\binom{B'}{A} \subset \binom{C}{A}$ is at most ε.*

Proof. Let $A, B \in \mathcal{K}(\mathbb{F}_q)$ be such that $A \leq B$, that is, $A = M_a(\mathbb{F}_q), B = M_b(\mathbb{F}_q)$, where a divides b. Let $k = \left|\binom{B}{A}\right|$. Fix ε.

Let $c > 64\varepsilon^{-2}\max\{\log(2k), \log(6\lceil\varepsilon^{-1}\rceil)\}$ be a multiple of b, and $C = M_c(\mathbb{F}_q)$. Now let γ be a continuous coloring of $\binom{C}{A}$. We seek to find some $B' \leq C$ with $B \cong B'$ such that the oscillation of γ on $\binom{B'}{A}$ is at most ε.

Let $\binom{B}{A} = \{A_1, \ldots, A_k\}$. For each $A_j \in \binom{B}{A}$, there exists some inner automorphism ϕ_j of B such that $\phi_j(A) = f_j A f_j^{-1} = A_j$, with f_j a unit of B, which can be taken to have determinant 1, so that $f_j \in SL_b(\mathbb{F}_q)$. Let $F = \{f_j \otimes 1_{c/b} : 1 \leq j \leq k\}$, so that $F \subset SL_c(\mathbb{F}_q)$. Now we define a coloring γ' of $SL_c(\mathbb{F}_q)$, given by $\gamma'^{-1})$.

Now let $m = \lceil \frac{3}{\varepsilon} \rceil$, and define $\mathcal{U}$ to be an open cover $\{U_i : 1 \leq i \leq m\}$ of $SL_c(\mathbb{F}_q)$, such that every $\frac{\varepsilon}{3}$-ball in $SL_c(\mathbb{F}_q)$ is contained in some U_i. Specifically, let us let $V_i = \left(\frac{i-2}{3}\varepsilon, \frac{i+1}{3}\varepsilon\right)$, observing that $\{V_i : 1 \leq i \leq m\}$ is an open cover for $[0,1]$. We note that the $\frac{\varepsilon}{3}$-ball around any point in $[0,1]$ is contained in some V_i. Then let $U_i = \gamma'^{-1}(V_i)$. As γ' is 1-Lipschitz, if $x \in SL_c(\mathbb{F}_q)$, and $B(x)$ is the $\frac{\varepsilon}{3}$-ball around it, then for any $y \in B(x)$, $d(\gamma'(y), \gamma'(x)) \leq d(y, x) < \varepsilon$, so $\gamma'(B(x))$ is contained in $B(\gamma'(x))$, which is contained in turn by V_i for some i, so $B(x) \subset \gamma'^{-1}(V_i) = U_i$.

Theorem 2.8 of Carderi and Thom's paper[4] states that there exists some $g \in SL_c(\mathbb{F}_q)$ such that $gF \subset U_i$ for some $1 \leq i \leq m$, as long as we take $c > 64\varepsilon^{-2}\max\{\log(2k), \log(2m)\}$, which is satisfied by our choice

$$c > 64\varepsilon^{-2}\max\{\log(2k), \log(6\lceil\varepsilon^{-1}\rceil)\}$$

Thus for each $f_j \in F$, $\gamma'(gf_j) \in V_i$, so $\gamma(gf_jAf_j^{-1}g^{-1}) \in V_i$. As $f_jAf_j^{-1} \subset B$, we have $gf_jAf_j^{-1}g^{-1} \subset gBg^{-1}$, and thus $gf_jAf_j^{-1}g^{-1} \in \binom{gBg^{-1}}{A}$. Thus if $S = \{gf_jAf_j^{-1}g^{-1} : 1 \leq j \leq k\}$, then $S \subset \binom{gBg^{-1}}{A}$, and $\gamma(S) \in V_i$, so the oscillation of γ on S is at most the diameter of V_i, which is ε. Let $1 \leq j, \ell \leq k$. If $gf_jAf_j^{-1}g^{-1} = gf_\ell A f_\ell^{-1}g^{-1}$, then $f_jAf_j^{-1} = f_\ell A f_\ell^{-1}$, so $j = \ell$. Thus each $gf_jAf_j^{-1}g^{-1}$ is distinct, and $\left|\{gf_jAf_j^{-1}g^{-1} : 1 \leq j \leq k\}\right| = k = \left|\binom{gBg^{-1}}{A}\right|$, so $\{gf_jAf_j^{-1}g^{-1} : 1 \leq j \leq k\} = \binom{gBg^{-1}}{A}$, and $\binom{gBg^{-1}}{A}$ has oscillation at most ε under γ, so we can let B'^{-1}. $\square$

Now to make precise our bound of $64\varepsilon^{-2}\max\{\log(2k), \log(6\lceil\varepsilon^{-1}\rceil)\}$, we must bound k. If $A' \in \binom{B}{A}$, and $\phi \in \mathrm{Aut}(B)$, then $\phi(A')$ will still be an embedded copy of A, and it is easy to see that $\phi : A' \mapsto \phi(A')$ defines a group action of $\mathrm{Aut}(B)$ on $\binom{B}{A}$. For any $A' \in \binom{B}{A}$, by the Skolem-Noether theorem there is some automorphism $\phi \in \mathrm{Aut}(B)$ such that $\phi(A \otimes 1) = A'$, so this action is transitive, and thus $k = \left|\binom{B}{A}\right| = \frac{|\mathrm{Aut}(B)|}{|\mathrm{Stab}(A\otimes 1)|} \leq |\mathrm{Aut}(B)|$. Thus we calculate $\mathrm{Aut}(B)$.

Theorem 6.2. $\mathrm{Aut}(M_n(\mathbb{F}_q)) \cong SL_n(\mathbb{F}_q)$

Proof. Define a map $\phi : GL_n(\mathbb{F}_q) \to \mathrm{Aut}(M_n(\mathbb{F}_q))$ given by $\phi(g) : x \mapsto gxg^{-1}$. Not only is this a group homomorphism, but it is onto, as the Skolem-Noether theorem guarantees that all automorphisms of $M_n(\mathbb{F}_q)$ are inner. Thus it suffices to determine the kernel of the map. $\ker\phi$ is exactly the center of $GL_n(\mathbb{F}_q)$, which is just the scalar multiples of the identity. Thus $\mathrm{Aut}(M_n(\mathbb{F}_q)) \cong GL_n(\mathbb{F}_q)/GL_1(\mathbb{F}_q) \cong SL_n(\mathbb{F}_q)$. □

As $|SL_n(\mathbb{F}_q)| = \frac{1}{q-1}\prod_{i=0}^{n-1}(q^n - q^i)$[1], $k \leq q^{b^2}$, so our Ramsey bound can be written as

$$64\varepsilon^{-2}(\log(2) + \max(b^2\log(q), \log(6\lceil\varepsilon^{-1}\rceil))$$

which is remarkably only quadratically dependent in b, and only slightly worse than quadratic in ε^{-1}.

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