# Distality in Continuous Logic

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# Metric Structures

## Definition

A *metric language* is just like a regular first-order language, but each symbol gets a Lipschitz constant.

## Definition

A *metric structure* consists of:

- A complete bounded metric space
- For each $n$-ary $k$-Lipschitz function symbol, a $k$-Lipschitz function $M^n \to M$
- For each $n$-ary $k$-Lipschitz relation symbol, a $k$-Lipschitz function $M^n \to [0, 1]$

# Formulas



## Definition

A *atomic formula* is defined as usual, except instead of $=$, the basic relation is $d(x, y)$.

## Definition

A *formula* is

- An atomic formula
- A continuous combination $u(\phi_1, \ldots, \phi_n)$ of formulas.
- $\sup_x \phi$ or $\inf_x \phi$

## Definition

A *definable predicate* is a uniform limit of formulas. This allows countably infinitely many variables.

# Indiscernible Sequences

Let $A \subseteq M$.

An $A$-*indiscernible sequence* is a sequence $(a_i : i \in I)$ such that

- $I$ is a linear order
- each $a_i \in M^x$ for some $x$
- for every $A$-formula $\phi(x_1, \ldots, x_n)$, $\phi(a_{i_1}, \ldots, a_{i_n})$ takes the same value for all $i_1 < \cdots < i_n$.

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# Fuzzy Set Systems



## Definition

A *fuzzy subset* of $X$ is a pair $(S_+, S_-)$ of disjoint subsets $S_+, S_- \subseteq X$.

Think $x \in S_+$ means "$x \in S$", $x \in S_-$ means "$x \notin S$", otherwise not sure.

# Fuzzy Set Systems

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## Definition

A *fuzzy set system* over $X$ is a set of fuzzy subsets of $X$.

# Fuzzy VC-Dimension

Let $\mathcal{F}$ be a fuzzy set system over $X$.

## Definition

- $\mathcal{F}$ *shatters* a finite set $Y \subseteq X$ when for every $Z \subseteq Y$, there is $(S_+, S_-) \in \mathcal{F}$ with $Z = Y \cap S_+$ and $Y \setminus Z = Y \cap S_-$.
- As usual, the *VC-dimension* of $\mathcal{F}$ is the largest $n$ such that $\mathcal{F}$ shatters a set of size $n$, or $\infty$ if there is no maximum.
- If the VC-dimension is *finite*, we say that $\mathcal{F}$ is a *VC-class*.

# VC Function Classes



## Definition

Given a function $\phi : X \times Y \to [0, 1]$, such as a formula, and $0 \leq r < s \leq 1$ define the fuzzy set system $\phi^Y_{r,s}$ to be all fuzzy sets $(\phi_{\leq r}(X; b), \phi_{\geq s}(X; b))$.

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We call $\phi$ a *VC-class* when $\phi^Y_{r,s}$ is always a fuzzy VC-class.

# Definitions of NIP



## Definition

A theory $T$ is NIP when any of the following equivalent conditions holds:

- For any indiscernible $a_0, a_1, a_2, \ldots,$

$$\phi(a_0; b), \phi(a_1; b), \phi(a_2; b), \ldots$$

  always converges.
- Every formula $\phi(x; y)$ defines a VC-class of sets/fuzzy sets/functions.
- Every formula $\phi(x; y)$ has an honest definition $\psi(x; z)$.

See Ben Yaacov, Chernikov-Simon

# $\varepsilon$-approximations



Fix

- $\phi(x; y)$ an NIP formula (VC-class)
- $\varepsilon > 0$
- $\mu$ a finitely-supported measure on $M^x$.

**Theorem (Essentially Alon, Ben-David, Cesa-Bianchi, Haussler)**

*There exists a tuple $(a_1, \ldots, a_n)$ with size $n \leq n(\phi, \varepsilon)$ that is a $\varepsilon$-approximation to $\phi$ with respect to $\mu$:*
*For all $b \in Y$,*

$$|\mathrm{Av}(a_1, \ldots, a_n; \phi(x; b)) - \mathbb{E}_\mu[\phi(x; b)]| \leq \varepsilon.$$

# $\varepsilon$-nets

Fix

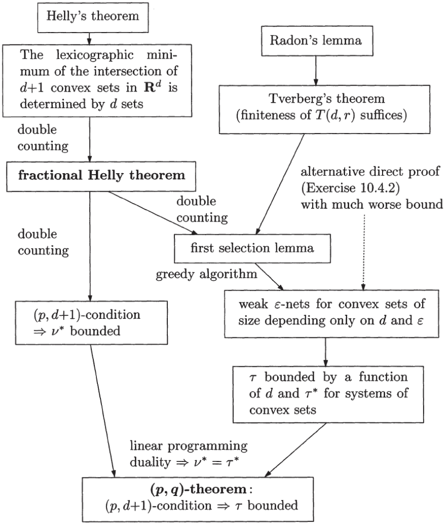
- $\phi(x; y)$
- $\varepsilon > 0$
- $\mu$ a finitely-supported measure on $M^x$
- $0 \leq r < s \leq 1$

## Theorem (A.)

*If $\phi_{r,s}^{M^y}$ has VC-dimension at most $d$, then there is a set $A$ of size at most $O\left(\frac{d}{\varepsilon} \ln \frac{1}{\varepsilon}\right)$ which is a $\varepsilon$-net for $\phi_{r,s}^{M^y}$ with respect to $\mu$:*
*For all $(S_+, S_-) \in \mathcal{F}$, if $\mu(S_+) \geq \varepsilon$, then $A \not\subseteq S_-$.*

# Fractional Helly Property and $(p, q)$-Theorem





# Honest Definitions



**Theorem (Chernikov-Simon, A.)**

*$M$ is NIP iff every formula $\phi(x; y)$ has an honest definition:*

*A definable predicate $\psi(x; z)$ such that for any closed set $A \subseteq M^x$ with $|A| \geq 2$ and $b \in M^y$, and any finite $A_0 \subseteq A$, there is a tuple $d$ in $A$ such that*

- *for all $a_0 \in A_0$, $\phi(a_0; b) = \psi(a_0; d)$*
- *for all $a \in A$, $\phi(a; b) \leq \psi(a; d)$.*

The proof that we can use the same $\psi$ for all $b$ and all $A_0$ uses the $(p, q)$-theorem.

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# Distal Structures



Distality is a strengthening of NIP, including

- (weakly, quasi-)$o$-minimal structures
- $\mathbb{Q}_p$.

A theory is distal when if

- $(a_i : i \in \mathbb{Q})$ is an indiscernible sequence
- $(a_i : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over $b$

then $(a_i : I \in \mathbb{Q})$ is indiscernible over $b$ also.

# Distal Cell Decompositions



Distal theories are also characterized by cell decompositions:

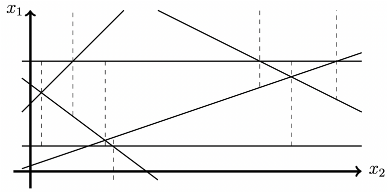


Figure: A discrete DCD for half-planes

For finite sets $B$, $M^x$ can be partitioned such that on each uniformly definable piece, $\mathrm{tp}_\phi(x/B)$ is controlled up to $\varepsilon$.

# Strong* Honest Definitions

## Theorem (Chernikov-Simon, A.)

*$M$ is distal iff every formula $\phi(x; y)$ has a strong* honest definition:*

*A definable predicate $\psi(x; z)$ such that for any closed set $A \subseteq M^x$ with $|A| \geq 2$ and $b \in M^y$, and any finite $A_0 \subseteq A$, there is a tuple $d$ in $A$ such that*

- *for all $a_0 \in A_0$, $\phi(a_0; b) = \psi(a_0; d)$*
- *for all $a \in M^x$, $\phi(a; b) \leq \psi(a; d)$.*

# Strong Honest Definitions

**Theorem (Chernikov-Simon, A.)**

*$M$ is distal iff every formula $\phi(x; y)$ has a strong honest definition:*

*A definable predicate $\theta(y; z)$ such that for any closed set $A \subseteq M^x$ with $|A| \geq 2$ and $b \in M^y$, for all finite $A_0 \subseteq A^x$, there is a tuple $d$ in $A$ with*

- $\theta(b; d) = 0$
- *for all* $b' \in M^y$, $a \in A_0$, $|\phi(a, b') - \phi(a, b)| \leq \theta(b'; d)$.

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# Examples of Distal Metric Structures

- Any discrete distal structure
- Dual linear continua
- Metric (weakly) *o*-minimal structures, such as
- Real closed metric valued fields (RCMVF)

Non-Examples:

- The randomization of *any* structure
- Expansions of Infinite-Dimensional Hilbert Spaces, Atomless Probability Algebras

# RCMVF

A model of RCMVF is the projective line of a real closed valued field with a nontrivial valuation in $\mathbb{R}^{\geq 0}$, and a corresponding metric.

**Theorem**

*RCMVF is distal.*

If we don't worry about the circularity of the order, it's weakly *o*-minimal. We can also prove distality with indiscernible sequences.

# Dual Linear Continua

If $L$ is a compact, connected linear order (a *linear continuum*),

- Let $M_L$ be the set of continuous nondecreasing surjections $L \to [0,1]$
- Give $M_L$ the sup metric
- For $\alpha \in [0,1]$, let $\phi_\alpha(f,g)$ be the value of $f$ when $f+g=\alpha$.

Then

- The type $\mathrm{tp}(f_1,\ldots,f_n)$ is defined by the image of $(f_1,\ldots,f_n)$ - a connected chain from 0 to 1 in $[0,1]^n$. These have the Hausdorff distance.
- The structure $M_{[0,1]}$ is $\aleph_0$-categorical, with $\mathrm{Aut}(M_{[0,1]}) = \mathrm{Homeo}^+([0,1])$.

# DCD Example: Dual Linear Continua



**Theorem (A., Ben Yaacov)**

*For each $\varepsilon > 0$, each $\phi_\alpha(x; y)$ admits an $\varepsilon$-distal cell decomposition, weakly defined by $\Psi$, where each $\psi(x; y_1, \ldots, y_k) \in \Psi$ has $k \leq \frac{4}{\varepsilon} + 2$.*

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*For each $\varepsilon > 0$, each $\phi_\alpha(x; y)$ admits an $\varepsilon$-distal cell decomposition, weakly defined by $\Psi$, where each $\psi(x; y_1, \ldots, y_k) \in \Psi$ has $k \leq \frac{4}{\varepsilon} + 2$.*

## Proof.

Fix $a \in M$, a finite $B \subset M_{[0,1]}$, and $\varepsilon > 0$. Let $n = \left\lfloor \frac{2}{\varepsilon} \right\rfloor$. Enough to determine whether $\phi_\alpha(a; b) < \frac{i}{n}$ for each $0 < i < n$ and each $b \in B$. It suffices to show that $\phi_\alpha(a; b) < \frac{i}{n}$ for one value of $b$ - maximizing $\sup\left(b^{-1}\left(\left\{\alpha - \frac{i}{n}\right\}\right)\right)$. $\square$

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# Keisler Measures



A *Keisler measure* is a regular Borel measure on a type space $S_x(A)$.

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A *generically stable* Keisler measure resembles a counting measure in an NIP structure, but these are more versatile for model-theoretic proofs.

Generically stable measures are particularly well-behaved (smooth) in distal structures - this characterizes distality.

# Definable Strong Erdős-Hajnal

## Theorem (Chernikov-Starchenko, A.)

*Fix $M$ distal, $\phi(x; y)$, and $\varepsilon > 0$.*
*There are $\delta > 0$ and $\psi^1(x; z_1), \psi^2(y; z_2)$ such that for any generically stable measures $\mu_1 \in \mathfrak{M}_x(M)$ and $\mu_2 \in \mathfrak{M}_y(M)$, there are*

- $c_1 \in M^{z_1}$ *and* $c_2 \in M^{z_2}$
- $\int \psi^1(x; c_1)\, d\mu_1 \geq \delta$ *and* $\int \psi^2(y; c_2)\, d\mu_2 \geq \delta$
- *The supports of* $\psi^1(x; c_1)$ *and* $\psi^2(y; c_2))$ *are* $(\phi, \varepsilon)$*-homogeneous.*

# Regularity

Definable strong Erdős-Hajnal is just one form of the *distal regularity lemma*:

For any $\varepsilon > 0$, by iteratively applying SEH, we can decompose both $M^{|x|}$ and $M^{|y|}$ into a bounded number of pieces, such that the total measure of the non-homogeneous "rectangles" is at most $\varepsilon$.

# Distal vs. SEH

Definable strong Erdős-Hajnal is equivalent to distality, but only a distal expansion is needed for strong Erdős-Hajnal on counting measures.

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Definable strong Erdős-Hajnal is equivalent to distality, but only a distal expansion is needed for strong Erdős-Hajnal on counting measures.

Does SEH for counting measures - or some other combinatorial property - imply a distal expansion?

Every known example of a non-distality comes from a failure of SEH, except...

# Nonexamples



A *Banach theory* is a metric structure expanding a Banach space.

**Theorem (Hanson)**

*Every Banach theory with infinite dimensional models has an infinite dimensional indiscernible subspace in some model. In particular, every such theory has an infinite indiscernible set, namely any orthonormal basis of an infinite dimensional indiscernible subspace.*

**Corollary**

*No Banach theory is distal, including IHS, probability algebras, and randomizations.*

Thank you, UMD!