Finding Order in Metric Structures

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March 28, 2025

Metric Structures

Definition

A *metric language* is just like a regular first-order language, consisting of functions and relations.

Definition

A *metric structure* consists of:

- A complete metric space of diameter 1
- For each n-ary function symbol, a uniformly continuous function $M^n \to M$
- For each n-ary relation symbol, a uniformly continuous function $M^n \to [0, 1]$

Formulas

Definition

An *atomic formula* is defined as usual, except instead of $=$, the basic relation is $d(x, y)$.

Definition

A *formula* is

- An atomic formula
- $u(\phi_1, \ldots, \phi_n)$ where ϕ_is are formulas and $u : [0,1]^n \to [0,1]$ is continuous
- $\sup_x \phi$ or $\inf_x \phi$

Definition

A *theory* is a set of *conditions* $\phi(x) = 0$.

Types and Definable Predicates

Definition

If $\bar{a} \in M^n$, the *type* $\mathrm{tp}(\bar{a})$ is the function $\phi \mapsto \phi(\bar{a})$.
The set of all types of $\bar{a} \in M^n$ in all models $M \vDash T$ is $S_n(T)$.

Fact (Compactness)

The space $S_n(T)$ is compact Hausdorff (in the coarsest topology making each formula continuous).

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Definition

A *definable predicate* is a continuous function $S_n \to [0, 1]$ - alternately, a uniform limit of formulas.

Basic Examples

Example

Let M be a boolean algebra with a probability measure μ. Can add

- metric $\mu(x \setminus y \cup y \setminus x)$
- functions $0, 1, ^c, \cap, \cup$
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Example

Let M be the unit ball of an infinite-dimensional Hilbert space, with the metric, $\langle \cdot, \cdot \rangle$, scalar multiplication, and partial addition.

Stability and Beyond

Those examples are *stable*: a well-studied class of structures (in classical and continuous logic) characterized by lacking definable orderings.

The many examples of stable metric structures give us a continuous logic analog to $(\mathbb{C}; 0, 1, +, \times)$. How do we find an "ordered" metric structure analogous to $(\mathbb{R}; 0, 1, +, \times, <)$?

Distal Structures

Distal structures are structures best understood in terms of a linear ordering:

- *o*-minimal structures such as $(\mathbb{Q}; <), (\mathbb{R}; <)$, $(\mathbb{R}; 0, 1, +, \times, <)$
- Weakly or quasi-*o*-minimal structures such as $(\mathbb{Z}; 0, 1, +, <)$
- The valued field $\mathbb{Q}_p$
- Some ordered differential fields of transseries.

Distal Structures

Distal structures can be defined many ways, in terms of:

- Indiscernible sequences (See bonus slides.)
- Definable cell decompositions
- Szemerédi regularity
- Keisler measures

My thesis extended these definitions, and some combinatorial consequences, to continuous logic.

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Theorem (Hanson)

No expansion of a probability algebra or infinite-dimensional Hilbert space is distal.

The proof uses the extreme amenability of the automorphism group to produce an indiscernible set.

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There are two other approaches, which do lead to distal metric structures:

- Dual Linear Continua (joint work with Itaï Ben Yaacov)
- Metric Linear Orders (ongoing joint work with Diego Bejarano) (if time allows)

Inspiration: Automorphism Groups

If M is an $\aleph_0$-categorical (metric) structure, then $G = \mathrm{Aut}(M)$ with the compact-open topology is a Polish group.

Fact (Ben Yaacov, Tsankov; Ibarlucía)

Certain model-theoretic properties of M are reflected in properties of G:

$$M \textit{ is stable} \iff \mathrm{RUC}(G) = \mathrm{WAP}(G)$$
$$M \textit{ is NIP} \iff \mathrm{RUC}(G) = \mathrm{Tame}(G)$$

There is no such characterization of distality (yet), but distality is close to “NIP and not stable”.

A Group Similar to $\mathrm{Aut}(\mathbb{Q},<)$

Fact (Megrelishvili, Pestov; see Ibarlucía)

The group $\mathrm{Aut}(\mathbb{Q},<)$ *is dense in* $\mathrm{Homeo}^+([0,1])$ - *the group of increasing self-homeomorphisms of* $[0,1]$.

- $\mathrm{RUC}(G) = \mathrm{Tame}(G)$ *for both groups*
- $\mathrm{WAP}(G) \subsetneq \mathrm{RUC}(G)$ *for both groups*

The takeaway is that if $\mathrm{Homeo}^+([0,1])$ is $\mathrm{Aut}(M)$ for some structure, then M is similar to $(\mathbb{Q},<)$, and likely distal.

Structures from Automorphism Groups

Fact (Melleray)

Any Polish group G is isomorphic to the automorphism group of some approximately ultrahomogeneous metric structure M.

To construct M from G, do the following:

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To construct M from G, do the following:

- Give G a left-invariant metric
- Let M be the metric completion of G
- Add distance relations to each orbit of G in M^n
- This makes each orbit closure a definable set.

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  - continuous
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- Add distance relations to orbits (G acts by precomposition)
- This language isn't great, but we can focus on the type spaces ($S_n \cong M^n/G$).

The Type Spaces

Theorem (A., Ben Yaacov)

Any type $\mathrm{tp}(f_1, \ldots, f_n)$ *is determined by the image of* $(f_1, \ldots, f_n) : [0,1] \to [0,1]^n$*, and these images are exactly the connected chains containing* 0 *and* 1*.*

The topology on the type space is given by the Hausdorff metric on compact subsets of $[0,1]^n$*.*

The type space is just M^n/G, and action by g on $(f_1, \ldots, f_n)$ reparametrizes, but does not change the image.

Reduction to 2 Variables

Theorem (A., Ben Yaacov)

Any type $\mathrm{tp}(f_1, \ldots, f_n)$ *is determined by the types* $\mathrm{tp}(f_i, f_j)$.

Corollary (A., Ben Yaacov)

$M_{[0,1]}$ *has quantifier elimination down to binary formulas.*

This works because any $f_i^{-1}(\{a\})$ is an interval, and if a family of n intervals intersects pairwise, they all intersect.

Another Language

- Binary formulas are continuous functions $S_2 \to [0, 1]$
- We just need enough symbols to uniquely determines types.

Definition

Let $\mathcal{L} = \{\phi_a : a \in \mathbb{Q} \cap [0, 2]\}$. Interpret these symbols so that $\phi_a(f, g)$ is the value of $f(x)$ when $f(x) + g(x) = a$.

This structure on $M_{[0,1]}$ is biïnterpretable with the original one, because they have the same type spaces.

Other Models

Let L be any *linear continuum*: a linear order which is equivalently

- compact and connected
- complete and dense

and assume L has distinct endpoints.

Then let M_L be the set of continuous nondecreasing surjections $L \to [0, 1]$, with the sup metric and the relations ϕ_a.

The elements of M_L realize the same types, so $M_L \equiv M_{[0,1]}$.

Other Models

Theorem (A., Ben Yaacov)

The models of $\mathrm{Th}(M_{[0,1]})$ are exactly the structures M_L where L is a linear continuum with distinct endpoints - call these dual linear continua.

If $M \equiv M_{[0,1]}$, and L is the chain in $[0,1]^M$ corresponding to the type of M itself, then M is isomorphic to M_L.

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- Itaï Ben Yaacov has described Ordered Real Closed Metric Valued Fields, but making the metric space bounded complicated things.
- Diego Bejarano and I are working to simplify this approach.

Metric Linear Orders

- Call M a *metric linear order* if
  - M has a bounded complete metric
  - M has a linear order
  - open balls are order-convex.
- M is a metric structure in the language $\{r\}$, with

$$r(x,y) = \begin{cases} 0 & x \leq y \\ d(x,y) & y \leq x \end{cases}$$

- Think of $r(x,y)$ as "the amount x is greater than y."

Axiomatizing Metric Linear Orders

Theorem (A., Bejarano)

Metric linear orders are axiomatized in $\{r\}$ by

- $\sup_{x,y} |(r(x,y) + r(y,x)) - d(x,y)| = 0$
  - $d(x,y) = r(x,y) + r(x,y)$
  - *Reflexivity*
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  - *Linearity/totality*
- $\sup_{x,y,z} r(x,z) \dot{-} (r(x,y) + r(y,z)) = 0$
  - *Triangle inequality*
  - *Transitivity*
  - *Monotonicity (look at $y \leq z \leq x$)*

Towards *o*-Minimality

Theorem (A., Bejarano)

If M is a metric linear order in the language $\{r\}$, TFAE:

- *every predicate $\phi(x)$ in one variable is a uniform limit of step functions*
- *every predicate $\phi(x)$ in one variable is quantifier-free definable.*

Theorem (A., Bejarano)

If M satisfies the above, it is distal.

Thank you, Rutgers!

Indiscernible Sequences

Definition

An *indiscernible sequence* in M^n is a sequence $(a_i : i \in I)$ where

- $a_i \in M^n$
- I is a linear order
- Whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$,

$$M \vDash \phi(a_{i_1}, \ldots, a_{i_n}) \iff M \vDash \phi(a_{j_1}, \ldots, a_{j_n})$$

Indiscernibles in DLO

Fact

In a dense linear order such as $(\mathbb{Q};<), (\mathbb{R};<)$, a sequence is indiscernible iff it is constant, increasing or decreasing.

Suppose we have a sequence $(a_i : i \in I)$, and after removing a or b, it is indiscernible. By monotonicity, it must be indiscernible with both a and b.

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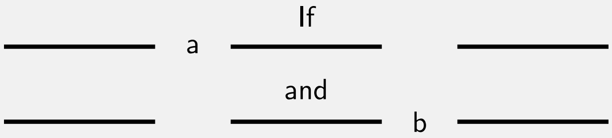

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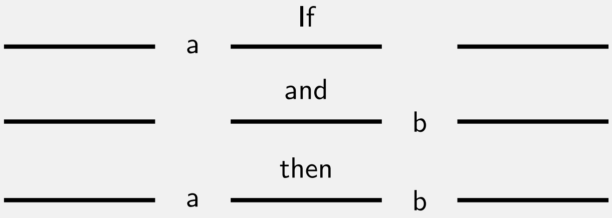

Definition of Distality

Definition

A structure M is *distal* when it is NIP and for every sequence $(a_i : i \in I)$, if after removing a or b, it is indiscernible, then it is indiscernible with a and b.

Indiscernibles in DLC

Let $(f_i : i \in I)$ be an indiscernible sequence in one variable in $M_{[0,1]}$.

Lemma (A., Ben Yaacov)

Any sequence $(a_i : i \in I)$ in the image of $(f_i : i \in I)$ is nondecreasing or nonincreasing.

Lemma (A., Ben Yaacov)

If $i < j$ and (a, b) is in the image of (f_i, f_j), then either (a, a) or (b, b) is also in that image.

Indiscernibles to Distality

- Let $(f_1, f_2, f_3, f_4, f_5)$ be such that removing f_2 or f_4 makes the sequence indiscernible.

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- By nonincreasing/nondecreasing property, $(a, a, a, b) \in \mathrm{im}(f_1, f_2, f_3, f_4)$, so $(a, b) \in \mathrm{im}(f_2, f_4)$.
- (Then we extrapolate to indiscernibles in $M^n_{[0,1]}$.)