Generically Stable Measures in Continuous Logic

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3 Orthogonality and Distality

Metric Structures

Definition

A *metric language* is just like a regular first-order language, but each symbol gets a Lipschitz constant.

Definition

A *metric structure* consists of:

- A complete bounded metric space
- For each n-ary k-Lipschitz function symbol, a k-Lipschitz function $M^n \to M$
- For each n-ary k-Lipschitz relation symbol, a k-Lipschitz function $M^n \to [0, 1]$

Formulas

Definition

A *atomic formula* is defined as usual, except instead of $=$, the basic relation is $d(x, y)$.

Definition

A *formula* is

- An atomic formula
- A continuous combination $u(\phi_1, \dots, \phi_n)$ of formulas.
- $\sup_x \phi$ or $\inf_x \phi$

Definition

A *definable predicate* is a uniform limit of formulas. This allows countably infinitely many variables.

Type Spaces

Definition

- A *complete type* in x over A is a consistent assignment of a real value to each formula $\phi(x)$ with parameters in A.
- Give the space $S_x(A)$ of such types the coarsest topology where $p \mapsto \phi(p)$ is continuous - this is compact Hausdorff.

Definable predicates are the continuous functions $S_x(A) \to [0, 1]$, and $p \mapsto \phi(p)$ is also continuous for them.

Properties of Types

Let $\mathcal{U}$ be a monster model, $p \in S_x(\mathcal{U})$.

Definition

- p is *A-invariant* when $\phi(p; b)$ depends only on $\mathrm{tp}(b/A)$.
- If p is A-invariant, define $F_{p,A}^{\phi} : S_y(A) \to [0, 1]$ by $F_{p,A}^{\phi}(q) = \phi(p; b)$ when $b \models q$.
- p is *A-Borel definable* when $F_{p,A}^{\phi}$ is Borel
- p is *A-definable* when $F_{p,A}^{\phi}$ is continuous
- p is *A-approximately realized/finitely satisfiable* when p is in the closure of A.

Keisler Measures

Definition

- A *Keisler measure* in x over A is a regular Borel probability measure on $S_x(A)$.
- Give the space $\mathfrak{M}_x(A)$ of such measures the coarsest topology where $\mu \mapsto \int_{S_x(A)} \phi(x)\, d\mu$ is continuous - also compact Hausdorff.

We could also define these in terms of linear functionals:

$$C(S_x(A), [0,1]) \to [0,1]$$
$$\phi(x) \mapsto \int_{S_x(A)} \phi(x)\, d\mu$$

Properties of Measures

Let $\mu \in \mathfrak{M}_x(\mathcal{U})$.

Definition

- μ is *A-invariant* when $\phi(p; b)$ depends only on $\mathrm{tp}(b/A)$.
- If μ is A-invariant, define $F^{\phi}_{\mu,A} : S_y(A) \to [0,1]$ by $F^{\phi}_{\mu,A}(q) = \int_{S_x(A)} \phi(x; b)\, d\mu$ when $b \models q$.
- μ is *A-Borel definable* when $F^{\phi}_{\mu,A}$ is Borel
- μ is *A-definable* when $F^{\phi}_{\mu,A}$ is continuous
- μ is *A-approximately realized/finitely satisfiable* when μ is in the closed convex hull of A.
- μ is *A-smooth* when it is the unique extension of $\mu|_A$.

Product Measures

Let $\mu \in \mathfrak{M}_x(M), \nu \in \mathfrak{M}_y(M)$.

Definition

$\omega \in \mathfrak{M}_{xy}(M)$ is a *product measure* of μ and ν when

$$\int_{S_{xy}(M)} \phi(x)\psi(y)\, d\omega = \int_{S_x(M)} \phi(x)\, d\mu \int_{S_y(M)} \psi(y)\, d\nu.$$

Definition

When μ is A-Borel definable, define $\mu \otimes \nu$ by

$$\int_{S_{xy}(M)} \phi(x;y)\, d(\mu \otimes \nu) = \int_{S_y(A')} F^{\phi}_{\mu,A'}\, d\nu|_{A'}$$

for $A' \supset A$ containing the parameters of $\phi(x;y)$.

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ε-approximations

Fix a family of functions $\mathcal{F} \subseteq [0,1]^X$, a probability measure μ on X, and $\varepsilon > 0$.

Definition

- Define the average

$$\mathrm{Av}(a_1, \ldots, a_n; f) = \frac{1}{n}(f(a_1) + \cdots + f(a_n)).$$

- Say that a tuple $(a_1, \ldots, a_n)$ is a ε-*approximation* for $\mathcal{F}$ with respect to μ when for every $f \in \mathcal{F}$,

$$|\mathrm{Av}(a_1, \ldots, a_n; f) - \mathbb{E}_\mu[f(x)]| \leq \varepsilon.$$

VC-Classes and NIP Theories

Definition

$\mathcal{F}$ is a *VC-class* when for every $\varepsilon > 0$, there is n such that for every μ, there is a ε-approximation of size $\leq n$.

There are other ways of characterizing these, including a generalization of VC-dimension.

Definition

T is NIP when every $\{\phi(x; b) : b \in \mathcal{U}^y\}$ is a VC-class.

fim and fam measures

Definition

- μ is *finitely approximated (fam)* in M when for every $\varphi(x; y) \in \mathcal{L}(M)$ and every $\varepsilon > 0$, there exists a ε-approximation $(a_1, \ldots, a_n) \in (M^x)^n$ for $\{\varphi(x; b) : b \in \mathcal{U}^y\}$ with respect to μ.
- μ is a *frequency interpretation measure (fim)* over M when for every $\varphi(x; y) \in \mathcal{L}(M)$, there is a family of formulas $(\theta_n(x_1, \ldots, x_n) : n \in \omega)$ with parameters in M such that $\lim_{n\to\infty} \mu^{(n)}(\theta_n(x_1, \ldots, x_n)) = 1$, and for every $\varepsilon > 0$, for large enough n, any $\bar{a} \in (\mathcal{U}^x)^n$ satisfying $\theta_n(\bar{a})$ is a ε-approximation to $\varphi(x; y)$ with respect to μ.

Generically Stable Measures

Assume NIP.

Theorem (A. generalizing Hrushovski-Pillay-Simon)

The following are equivalent for $\mu \in \mathfrak{M}_x(M)$:

- μ *is definable and finitely satisfiable*
- μ *is fam*
- μ *is fim*
- $\mu(x) \otimes \mu(y) = \mu(y) \otimes \mu(x)$

Definition

$\mu \in \mathfrak{M}_x(M)$ is *generically stable* when these conditions hold.

Approximating NIP Predicates

Using the fim property, we can show the following:

Theorem (A. generalizing Chernikov-Starchenko)

Assume NIP. For each $\phi(x; y)$ and $\varepsilon > 0$, there are θ_i, χ_i with parameters in M such that for all generically stable measures $\mu \in \mathfrak{M}_x(M), \nu \in \mathfrak{M}_y(M)$,

$$\int_{S_{xy}(M)} \left| \phi(x; y) - \sum_{i=1}^{n} \theta_i(x)\chi_i(y) \right| \, d\mu \otimes \nu.$$

Regularity Partitions

- For each $\theta_i(x)$, make a partition of unity $\sum_{\psi \in \Psi} \psi(x) = 1$
- On each support $\{\psi(x) > 0\}$, $\theta_i(x)$ varies by at most ε
- Each ψ is definable, or each is constructible (indicator of difference of zerosets)
- Combine these partitions to get a partition that works for all θ_i
- Do the same for χ_i
- Taking products from the two partitions, we get a partition on $M^x \times M^y$

Regularity Partitions

Lemma (A.)

For any $\phi(x;y) = \sum_{i=1}^{n} \theta_i(x)\chi_i(y)$, there is a partition of unity $\sum_{\psi \in \Psi} \psi(x;y) = 1$ where

- *Each $\psi(x;y)$ factors as $\psi_x(x)\psi_y(y)$*
- *Each ψ is definable, or each ψ is constructible*
- *The support of each $\psi(x;y)$ is (ϕ, ε)-homogeneous (the value of ϕ varies by at most ε)*

NIP Regularity

Theorem (A., after a similar result by Chernikov-Towsner)

Assume NIP.
For any $\phi(x;y)$, $\varepsilon > 0$, μ,ν generically stable, there is a partition of unity $\sum_{\psi\in\Psi}\psi(x;y) = 1$ where

- *Each $\psi(x;y)$ factors as $\psi_x(x)\psi_y(y)$*
- *Each ψ is definable, or each ψ is constructible, only parameters depend on μ,ν*
- *For each ψ, there is r_ψ such that*

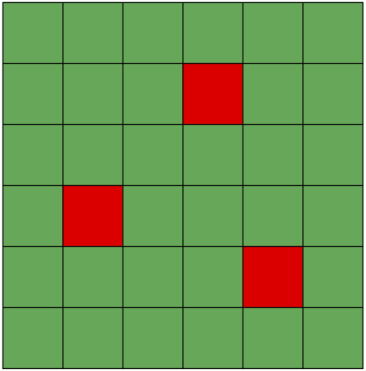
$$\sum_{\psi}\int_{S_{xy}(M)}\psi(x;y)\left|\phi(x;y) - r_\phi\right|\, d\mu\otimes\nu \le \varepsilon$$

NIP Regularity

Call ψ good if

$$\int_{S_{xy}(M)} \psi(x;y)\,|\phi(x;y) - r_\phi|\; d\mu\otimes\nu \leq \varepsilon \int_{S_{xy}(M)} \psi(x;y)|\; d\mu\otimes\nu$$

Can get total integral of bad ψs below δ.



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Orthogonal Measures

Definition

$\mu \in \mathfrak{M}_x(M), \nu \in \mathfrak{M}_y(M)$ are *weakly orthogonal* when they have a unique product measure.

Equivalently, the integration functional

$$\sum_{i=1}^{n} \theta_i(x)\chi_i(y) \mapsto \sum_{i=1}^{n} \left(\int_{S_x(M)} \theta_i(x)\, d\mu\right)\left(\int_{S_y(M)} \chi_i(y)\, d\nu\right)$$

has a unique extension.

Extensions of Keisler Functionals

Lemma (A.)

If V is a vector subspace of the definable predicates on $S_x(M)$, and μ a functional/partial measure defined on V, then we can find an extension μ' with

$$\int_{S_x(M)} \phi(x)\, d\mu' = r$$

if and only if

$$\int_{S_x(M)} \psi^-(x)\, d\mu \leq r \leq \int_{S_x(M)} \psi^+(x)\, d\mu$$

for all $\psi^-(x) \leq \phi(x) \leq \psi^+(x)$.

Orthogonal Measures

Lemma (A. generalizing Simon)

Measures $\mu \in \mathfrak{M}_x(M), \nu \in \mathfrak{M}_y(M)$ *are weakly orthogonal if and only if for every* $\phi(x; y)$ *and* $\varepsilon > 0$, *there exist* $\psi^-(x; y), \psi^+(x; y)$, *each of the form* $\sum_{i=1}^m \theta_i^\pm(x)\chi_i^\pm(y)$, *such that*

- $\psi^-(x; y) \leq \phi(x; y) \leq \psi^+(x; y)$.
- *For any product measure* ω *of* μ, ν, $\int_{S_{xy}(M)} (\psi^+ - \psi^-) \, d\omega \leq \varepsilon$.

Orthogonal Regularity

Theorem (A., generalizing Simon)

For any $\phi(x; y)$, $\varepsilon, \delta > 0$, μ, ν weakly orthogonal, there is a partition of unity $\sum_{\psi \in \Psi} \psi(x; y) = 1$ where

- *Each $\psi(x; y)$ factors as $\psi_x(x)\psi_y(y)$*
- *Each ψ is definable, or each ψ is constructible*
- *The total integral of non-(ϕ, ε)-homogeneous ψ is at most δ*

Orthogonal Regularity

Theorem (A., generalizing Simon)

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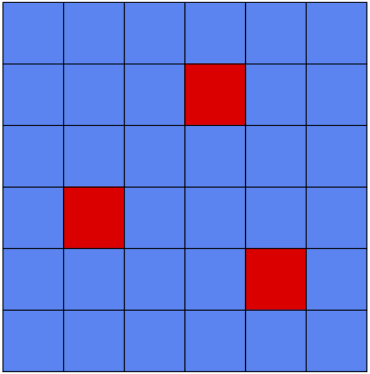
Orthogonal Regularity

Good pieces are (ϕ, ε)-homogeneous
Bad pieces have total integral $\leq \delta$



Distal (Metric) Structures

Theorem (A.)

For an NIP theory T, the following are equivalent:

- *Every generically stable measure is smooth*
- *Any two generically stable measures commute.*

We call such a theory, and its models, *distal*.
For example:

- o-minimal structures such as RCF
- $\mathbb{Q}_p$ with ring or linear structure (in Macintyre-ish languages)

Metric structures:

- Real closed metric valued fields
- Dual linear continua

Genericaly Stable and Smooth

Lemma (A. generalizing Simon)

μ is smooth over M if and only if it's weakly orthogonal to all measures over M.

This gives us orthogonal regularity for generically stable measures, which we can uniformize using

Lemma (A. generalizing Simon)

The ultraproduct of a family of generically stable measures is generically stable.

Distal Regularity

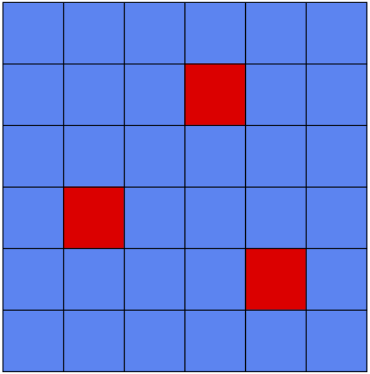
Theorem (A., generalizing Chernikov-Starchenko)

Assume T distal. For any $\phi(x;y)$, $\varepsilon, \delta > 0$, μ, ν generically stable, there is a partition of unity $\sum_{\psi \in \Psi} \psi(x;y) = 1$ where

- *Each $\psi(x;y)$ factors as $\psi_x(x)\psi_y(y)$*
- *Each ψ is definable, or each ψ is constructible, only parameters depend on μ, ν*
- *The total integral of non-(ϕ, ε)-homogeneous ψ is at most δ*

Distal Regularity

Same as orthogonal regularity, but uniformly definable pieces.



Definable Strong Erdős-Hajnal

Theorem

Assume T distal. For any $\phi(x;y)$, $\varepsilon > 0$, there are $\theta(x;w), \chi(y;z), \delta > 0$ such that for all μ, ν generically stable, there are $c \in M^w, d \in M^z$ where

- $\theta(x;w)\chi(y;z) > 0$ *is* (ϕ, ε)*-homogeneous*
- $\int_{S_x(M)} \theta(x;c)\, d\mu \geq \delta$
- $\int_{S_y(M)} \chi(y;d)\, d\nu \geq \delta$

Thank you, PKU!