Finding Order in Metric Structures

Aaron Anderson (joint with Diego Bejarano)

UPenn

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Metric Structures

Definition

A *metric language* is just like a regular first-order language, consisting of functions and relations.

Definition

A *metric structure* consists of:

- A complete metric space of diameter ≤ 1
- For each n-ary function symbol, a uniformly continuous function $M^n \to M$
- For each n-ary relation symbol, a uniformly continuous function $M^n \to [0, 1]$

Formulas

Definition

An *atomic formula* is defined as usual, except instead of $=$, the basic relation is $d(x, y)$.

Definition

A *formula* is

- An atomic formula
- $u(\phi_1, \ldots, \phi_n)$ where ϕ_is are formulas and $u : [0,1]^n \to [0,1]$ is continuous
- $\sup_x \phi$ or $\inf_x \phi$

Making Linear Orders Metric

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- Itaï Ben Yaacov has described Ordered Real Closed Metric Valued Fields, but making the metric space bounded complicated things.
- Diego Bejarano and I are working to simplify this approach.

Metric Linear Orders

- Call M a *metric linear order* if
  - M has a complete metric of diameter ≤ 1
  - M has a linear order
  - open balls are order-convex.
- M is a metric structure in the language $\{r\}$, with

$$r(x,y) = \begin{cases} 0 & x \leq y \\ d(x,y) & y \leq x \end{cases}$$

- Think of $r(x,y)$ as "the amount x is greater than y," or

$$r(x,y) = d\left(x, (\infty, y]\right).$$

Axiomatizing Metric Linear Orders

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Metric linear orders are axiomatized in $\{r\}$ *by*

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- $\sup_{x,y} \min\{r(x,y), r(y,x)\} = 0$
- $\sup_{x,y,z} r(x,z) \dot{-} (r(x,y) + r(y,z)) = 0$

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- DLO and $\mathrm{Th}(\mathbb{Z}, <)$ are two useful completions of the theory of linear orders
- We seek analogous completions of MLO
- For simplicity, assume the metric is an ultrametric.

Axiomatizing Ultrametric Dense Linear Orders

Definition

Let UDLO be the theory of *ultrametric-dense linear orders*, consisting of MLO with the following axioms:

- $d(x,z) \leq \max(d(x,y), d(y,z))$
- For any rational $p \in \mathbb{Q} \cap [0,1]$, $\sup_x \inf_y |r(x,y) - p| = 0$
- For any rational $p \in \mathbb{Q} \cap [0,1]$, $\sup_x \inf_y |r(y,x) - p| = 0$.

Basically, the distances from x to $y > x$ are dense in $[0,1]$.

Stable Diversion: Dense Ultrametrics

Call an ultrametric space *dense* if the set of distances to any point is dense in $[0, 1]$.

Fact (Conant)

The theory of dense ultrametrics is complete and has QE, but is not $\aleph_0$-categorical.

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The theory of dense ultrametrics is the model companion of the theory of ultrametrics, and is approximately $\aleph_0$-categorical.

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Theorem (A., Bejarano)

In a model of UDLO,

- *the metric and order topologies agree*
- dcl *is closure*

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Lemma (A., Bejarano)

$U_S \models$ UDLO.

Theorem (Van Thé)

U_S has extremely amenable automorphism group, following from Fraïssé theory in the **discrete-logic** *language of ordered S-valued metric spaces.*

o-Minimality in Discrete Logic

Fact

If M expands a linear order, TFAE:

- *every formula $\phi(x)$ in one variable is qf-definable in $\{<\}$*
- *every formula $\phi(x)$ in one variable is a finite union of intervals.*

- If these happen, M is *o-minimal*.

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- If these happen, M is *o-minimal*.
- How do we describe these properties for MLOs?

Metric *o*-Minimality

Theorem (A., Bejarano)

If M expands a metric linear order, TFAE:

- *every formula $\phi(x)$ in one variable is qf-definable in $\{r\}$*
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By QE, a model of UDLO is o-minimal.

Regulated Functions

Regulated functions $[a, b] \to \mathbb{R}$ were defined by Bourbaki.

Lemma (A. Bejarano)

If M is a linear order, $f : M \to [0, 1]$, TFAE:

1. *f is a uniform limit of step functions*
2. *For any $a < b$, M can be partitioned into finitely many intervals on which either $f(x) > a$ or $f(x) < b$.*

If $M \models \mathrm{UDLO}$, and $f : M \to M$ is definable ($d(f(x), y)$ is a formula), then f is continuous and satisfies (2).

Metric Valued Fields

Definition

A *metric valued field* is a field K with an absolute value $|\cdot| : K \to \mathbb{R}^{\geq 0}$ satisfying valuation axioms:

- $|x| = 0 \iff x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq \max(|x|, |y|)$

which gives rise to an ultrametric $d(x, y) = |x - y|$.

Real Closed Metric Valued Fields

Definition

If K is a metric valued field equipped with an order, $K \equiv \mathbb{R}$ as ordered fields in classical logic, and $|\cdot|$ takes values outside $\{0, 1\}$, call K a *real closed metric valued field*.

These are almost metric structures, but the metric is unbounded, a problem for continuous logic.

Metric Valued Fields, à Rideau-Kikuchi, Scanlon, Simon

- Typically we just deal with the ball $B_{\leq 1}(0)$.
- For a metric valued field K, $\{x \in K : |x| \leq 1\}$ is a subring - the *valuation ring*.
- This is a metric structure in the ring language.

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- For a metric valued field K, $\{x \in K : |x| \leq 1\}$ is a subring - the *valuation ring*.
- This is a metric structure in the ring language.
- Unfortunately, these are elementarily equivalent to other convex subrings of metric valued fields.

Ordered Real Closed Metric Valuation Rings

Theorem (A., Bejarano)

- *Models of our theory* ORCMVR *are convex subrings of real closed metric valued fields.*
- *These have quantifier-elimination once we add a divisibility relation.*
- *These can only be weakly o-minimal.*

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Theorem (A., Bejarano)

Any model of ORCMVF *(the projective line) is an o-minimal expansion of a model of* UDCO*, the complete theory of ultrametric dense cyclic orders.*

Thank you, UMD!