Triangulated Polygons and Frieze Patterns 1

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1 Symmetries of Frieze Patterns

First, a lesson in architecture. A *frieze* is a horizontal band of decoration around a building. These are some friezes from buildings in Ancient Persepolis. Note how they feature symmetrical, repeating patterns, and it is easy to see how you could keep drawing them to the left or the right, infinitely far. In mathematics, a *frieze pattern* will be a pattern drawn on an infinitely long band (if you want some kind of precision, imagine the set $\mathbb{R} \times [0, 1]$, which is the set of points (x, y) in the plane where $0 \leq y \leq 1$), that has a certain kind of symmetry.

Figure 1: Friezes from Ancient Persepolis

The symmetry groups of frieze patterns are called *frieze groups*, and there are only 7 of them (up to isomorphism). They are given by the following examples using common symbols:

LLLLLLLL
LΓLΓLΓLΓ
VVVVVVVV
SSSSSSSS
VΛVΛVΛVΛ
BBBBBBBB
HHHHHHHH

Problem 1 These symmetry groups include the following types of symmetries: horizontal reflections (H), vertical reflections (V), (horizontal) translations (T), (180°) rotations (R), and glide reflections (G). Using those abbreviations, can you identify which types of symmetry are present in each of the 7 frieze groups?

Problem 2 Which symmetry group(s) do the frieze patterns in Figure 1 have?

Problem 3 Can you classify the symmetry groups of the following tables of Roman numerals:

```
(a) ... I   I   I   I ...
```

```
(b)     I    I    I    I ...
    ...  I    I    I    I
```

```
(c)      I     I      I     I ...
    ...  I     II     I     II
         I     I      I     I ...
```

```
(d)      I  I  I  I  I  I
    ... III I  II II I  III
         II II I  III I  II ...
    ...  I  I  I  I  I  I
```

Figure 2: Frieze Patterns of Roman Numerals

2 Unimodular Frieze Patterns of Numbers

In the last problem, the Roman numerals I, II, III all conveniently had lots of symmetry. However, as this is a *math* circle, we'd like to study the patterns of numbers, and we care more about arithmetic properties than the way the numbers are drawn, so we'll use regular Arabic numerals, and just look at the symmetries of their arrangement.

Our new, numerical frieze patterns will be a pattern made of numbers, arranged in (infinite) rows, with each row staggered a bit from the last. The first (top) row will be all 1s, as will the last row. We fill the rows in between with any numbers, which are positive unless specified, subject to one more constraint, which we'll get to in a moment.

```
        1   1   1   1   1   1
    ...   1   3   1   3   1   ...
        2   2   2   2   2   2
    ...   3   1   3   1   3   ...
        1   1   1   1   1   1
```

```
        1   1   1   1   1   1
    ...   1   2   3   1   2   ...
        2   1   5   2   1   5
    ...   1   2   3   1   2   ...
        1   1   1   1   1   1
```

```
      1   1   1   1   1   1   1   1
  ...   1   3   2   2   1   4   2   ...
      1   2   5   3   1   3   7   1
  ...   1   3   7   1   2   5   3   ...
      2   1   4   2   1   3   2   2
  ...   1   1   1   1   1   1   1   ...
```

Figure 3: Frieze Patterns of Numbers

We will also define the *order* of a frieze pattern of numbers to be 1 more than the number of rows, so a frieze pattern of order n has $n-1$ rows, the first and last of which are all 1s.

Problem 4 Our additional constraint, satisfied by these examples, is a property of "diamonds" of four numbers. Any time we have numbers a, b, c, d arranged in the following shape, they must satisfy a simple equation. Can you guess what it is? A number frieze with this property is called *unimodular*, and for the rest of the worksheet, we will only care about unimodular number friezes.

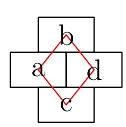

Figure 4: Unimodular condition

2.1 Periodicity in Order 5

Problem 5 Gauss proved the following identity:

$$u_0\left((1+u_4-u_1u_2)-(1+u_1-u_3u_4)\right)=(1+u_2)(1+u_3-u_0u_1)-(1+u_3)(1+u_2-u_4u_1)$$

Use it to show $u_0 = u_5$ in the following frieze pattern, and then describe the rest of the frieze pattern:

1	1	1	1	1
u_0	u_1	u_2		
	u_3	u_4	u_5	
1	1	1	1	1

Figure 5

2.2 Constant rows

Let's think for a while about how to build unimodular frieze patterns where each row only consists of one number, repeating.

Problem 6

(a) In a unimodular frieze pattern of order 4, the first and third row are all ones. If the remaining row is also only one number, call it x, repeated, what possible values can x take?

(b) In a unimodular frieze pattern of order 5, there are now two rows to consider between the 1s. Say every entry in the second row is x, and the third row is all y. What can x and y be? How do they have to be related?

Problem 7 Recall this theorem from Euclidean Geometry: If $ABCD$ is a polygon inscribed in a circle, then

$$AB \times CD + BC \times DA = AC \times BD$$

(a) If $ABCDE$ is a regular pentagon with all side lengths 1, can you use that identity to build a frieze pattern where each row consists of only one number, and each such number is the length of a side or diagonal of $ABCDE$?

(b) Can you use this logic to build a frieze pattern of order n for any $n > 5$, still keeping each row constant? How unique is it?

3 Useful Patterns and Formulas

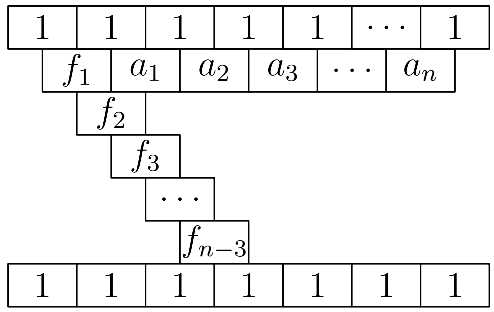

Figure 6: Selected elements of a frieze pattern of order n

Problem 8

(a) Say for now that $n = 8$.

In the above picture, say the diagonal is filled in, that is, $f_1, \ldots, f_5$ are fixed. If we assume this can be extended to *some* frieze pattern, is the rest of the frieze pattern determined uniquely? Can you find formulas for $a_1, \ldots, a_5$ in terms of $f_1, \ldots, f_5$?

(b) In that same picture, say that $a_1, \ldots, a_5$ are fixed. Is the rest of the frieze pattern determined uniquely? Can you find formulas for $f_1, \ldots, f_5$ in terms of $a_1, \ldots, a_5$?

Problem 9 Assume $n \geq 4$. Defining $a_1, \ldots, a_n$ as above, prove that $a_r a_{r-1} > 1$.

Problem 10 We define the *period* of a frieze pattern to be the least positive integer p such that each row repeats every p numbers. In particular, in a pattern of period p, $a_k = a_{k+p}$. The period of a frieze pattern of order n has period p dividing n, and you should assume this for now, but we will not prove it yet.

(a) Determine the period of each frieze pattern in Figure 2.

(b) In Section 2.1, we saw that all frieze patterns of order 5 have either period 5 or period 1. Verify that in the examples from part (a), the period divides the order of the frieze pattern.

Problem 11 For what values of n does the period of an order n frieze pattern have to be strictly less than n?