Logic 2 Homework 4

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Feel free to cite results from your lecture notes, the official lecture notes, or textbooks referenced on the course website. You are also allowed and encouraged to collaborate with your classmates, but you must write your own solutions. Please do not use any other sources without first discussing with the instructor.

Problem 4.1. A cell is an "open cell" if and only if it is an open set, justifying the notation.

Problem 4.2. Finish the proof of (Lemma 4.6.2) for arbitrary n.
Specifically, assuming PC_n, prove that if $D \subseteq M^{n+1}$ is definable, TFAE:

1. D is sparse
2. The set D' of $\bar{a} \in M^n$ with $D_{\bar{a}}$ infinite is sparse
3. D is nowhere dense (its closure is sparse).

Exercises from van den Dries:

Problem 4.3 (Exercise (2.19) 5)**.** Fix an o-minimal structure $\mathcal{M}$.
Let $X_1, \ldots, X_k \subseteq M^n$ be distinct nonempty definably connected sets, and let $X = \bigcup_{i=1}^{k} X_i$. Define a graph on vertices $\{1, \ldots, k\}$ by drawing an edge between i and j if $\mathrm{cl}(X_i) \cap X_j \neq \emptyset$ or $\mathrm{cl}(X_j) \cap X_i \neq \emptyset$.
Show that the connected components of this graph correspond to definably connected components of X in the following way: If $S \subseteq \{1, \ldots, k\}$ is a connected component, then $\bigcup_{i \in S} X_i$ is definably connected, and if $S \subseteq \{1, \ldots, k\}$ is disconnected, then $\bigcup_{i \in S} X_i$ is not definably connected.

Problem 4.4 (Exercise (2.19) 6)**.** Suppose $\mathcal{M}$ is an o-minimal structure, and $\mathcal{M}'$ is an o-minimal expansion of $\mathcal{M}$. (A structure in a language with possibly more relation and function symbols.)
Show that if $D \subseteq M^n$ is definable in $\mathcal{M}$, then D is definably connected as a definable set over $\mathcal{M}$ if and only if it is definably connected as a definable set over $\mathcal{M}'$.

Problem 4.5 (Exercise (2.19) 7)**.** Suppose $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}; <)$. Show that if $D \subseteq M^n$ is definable, then D is definably connected if and only if it is connected.